Interest Rate Derivatives: Lecture notes

Dr. Graeme West
Financial Modelling Agency
graeme@finmod.co.za
www.finmod.co.za

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Chapter 1

Introduction to Interest Rates

1.1 Introduction

In equity option pricing we make the assumption throughout that interest rates were previsible. This greatly simplifies discussions of hedging and replication, and allows the use of the Black-Scholes analysis and framework. The major implication of this assumption concerns the numeraire in the martingale pricing formula. Recall that the chosen numeraire is the money-market account (continuous investment/borrowing at deterministic interest rates). On application of the assumption, we could bring the numeraire $B(T)$ in the martingale pricing formula: $\frac{f(0)}{B(0)} = \mathbb{E}_0^{\mathbb{Q}}\left[\frac{f(T)}{B(T)}\right]$[1] out of the expectation to evaluate the price of the derivative as: $f(0) = \frac{B(0)}{B(T)}\mathbb{E}_0^{\mathbb{Q}}[f(T)]$.

This equation forms the basis of the Black-Scholes solution. The measure Q is the so-called Risk Neutral Measure, is unique, and implies that the stochastic processes for the underlying (default-free) securities and their derivatives have an expected drift rate of r (the prevailing risk-free rate), under Q. (In fact, under Q, ALL traded securities have an expected return of r!)

Despite our reservations, the model is extensively used, as seen for example in the extensive use of the Black-Scholes formula for short-dated options. This is a consequence of the underlying stochastic process (for the stock) being fairly remote from the stochastic process for the interest rate.

[1] Some notation: $B(T)$ is the amount of money that 1 at time 0 has grown to by time T, by continual reinvestment at the short dated rate: $B(\cdot)$ is called the money market account, or simply the bank account. Usually this short dated rate is taken to be the overnight rate, so money is placed in the overnight account, and left there, compounding each day. Of course, by this, we mean the usual international meaning of the overnight account, in South Africa, a true overnight rate does not exist, as discussed in West [2009].

$Z(t,T)$ is the discount factor for time T as observed at time t. It is known when we reach t, from the bootstrap of the yield curve at that time. Often, $t = 0$, in which case we might abbreviate $Z(0,T)$ to $Z(T)$.

$Z(0;t,T)$ is the forward discount factor from time t to time T. As proved in West [2009], it is equal to $\frac{Z(0,T)}{Z(0,t)}$.

$C(t,T)$ is the capitalisation factor for time T as observed at time t. The same comments apply. $C(0;t,T)$ is the forward capitalisation factor from time t to time T. As proved in West [2009], it is equal to $\frac{C(0,T)}{C(0,t)}$. Note that $Z(\cdot)$ and $C(\cdot)$ are inverse.

We do not know the value of $B(T)$ at time 0 because interest rates are stochastic. When we price equity derivatives, we assume that this stochasticity is removed, so $B(\cdot) = C(\cdot)$.

However, this cannot be the case when options and derivatives are written either on interest rates, or on securities whose values are dependent on interest rates (e.g. bond options, swaps, caps, floors etc.) In these cases, it is exactly the fluctuation of interest rates that the option buyer seeks to hedge; so an assumption of constancy of interest rates makes little sense. In particular, the bond option model we saw in West [2009] is very problematic.

As we saw in West [2009] vanilla type interest rate derivatives such as deposits (JIBAR deposits), FRAs and swaps do not require interest rate modelling, as they are priced using pure no arbitrage considerations. They can be replicated using instruments that pay certain cash flows in the future and consequently require no statistical modelling at all. They do require an accurate yield curve, though. Producing this is not always as simple a task as it would appear. We saw a (naively) simple method in West [2009]. For more information on the subtle difficulties that can arise, see Hagan and West [2006], Hagan and West [2008].

It will be assumed that you are perfectly familiar with all of the interest rate material dealt with in West [2009].

In order to price more complicated interest rate products that include optionality, we need to arrive at a statistical model of the evolution of the yield curve. The models we will consider in this course model all interest rates as dependent on a single rate, often termed the short rate. The evolution of the short rate then governs the evolution of all rates along the entire curve. However, this is quite a task, as a change in the yield curve is a complicated phenomenon, since it may undergo combinations of parallel shifts, slope changes and curvature changes.

These models are roughly divided into two categories: equilibrium models and no arbitrage models. In the equilibrium approach, if the model can be trusted to give a fundamentally correct, albeit necessarily simplified, description of economic reality then there will be discrepancies between model and market values. According to the model these discrepancies will reflect trading opportunities! Equilibrium Models attempt to describe the economy of interest rates as a whole. Clearly this approach is quiet abstract, and is not much used - although there are some equilibrium models that can be reformulated as no arbitrage models.

The pure no arbitrage approach seeks to represent the value of a complicated interest rate derivative in terms of vanilla instruments or cash instruments. The prices of these vanilla instruments will be taken as given and any model must recover their actual traded prices. No Arbitrage Models take the market prices of vanilla products as basic building blocks, and infer from them more complicated derivative prices. However, different models could give different prices of the more complicated instruments, even though they recover the same prices for the vanilla instruments!

Another complicating feature is that we cannot buy and sell interest rates. THE UNDERLYING OF THE MODEL IS NOT A TRADEABLE INSTRUMENT. The construction of a “replicating” portfolio requires more thought: it isn’t just the delta and cash, as it is for equity: both of these factors don’t perform as we would naively like. What we can buy and sell is bonds, whose prices are themselves derivative of interest rates.

Initially, we examine default-free securities and the term structure of interest rates. Once we have done this, we are in a position to model the movements of the yield curve.

1.2 Day count conventions

Denote the generic period between two payment dates as α parts of a year. The rules could be different for bonds and swaps, and even for the floating and fixed legs of the swaps. Moreover, the rules differ by jurisdiction. The relevant markets are

- The bond market: the market for the issuing of treasury bonds. Day count conventions are relevant for the accrual of interest and hence the conversions between clean and dirty price.[2]
- The money market: the market for FRAs and hence the market for the floating leg of swaps.
- The swap market: the market for the fixed leg of swaps.

The day count conventions in the various markets are as follows

	spot/value date	bond	money	fixed swap
RSA	t, $t+3$	Actual/365	Actual/365	Actual/365
USA	$t+2$	Actual/Actual	Actual/360	30/360 [3]
UK	t	Actual/Actual	Actual/365	Actual/365
Euro	$t+2$	Actual/Actual	Actual/360	30/360
Japan	$t+2$	Actual/365	Actual/360	Actual/365
Canada	t	Actual/365	Actual/365	Actual/365
Australia	t	Actual/Actual	Actual/365	Actual/365

where Actual/Actual appears in International Swaps and Derivatives Association [25 November 1998]. This is rather tricky, they point out that at the time there were three conflicting interpretations of this rule. The approach taken thus needs to be recorded when a deal is made. The approaches are a follows:

(i) ISDA Actual/actual (historical). Split the period of interest into the years in which it occurs. For each year, divide the number of actual days in the period by the number of days in that year. Day count is equal to the sum of these fractions.

For example, if the period is from 20 September 2003 to 20 March 2004, and we are now at 13 January 2004, then the period day count is $\frac{102}{365}+\frac{80}{366}$ and the time elapsed day count is $\frac{102}{365}+\frac{13}{366}$.

(ii) AFB Actual/Actual Euro. The numerator is the actual number of days, the denominator is either 365 or 366 depending on whether or not the period includes a 29 February.

In the above example, the period day count is $\frac{182}{366}$ and the elapsed day count is $\frac{115}{366}$.

(iii) ISMA Actual/Actual Bond. This is the actual number of days, divided by the product of the number of days in the period and the number of periods in the year.

[2] In fact it is unlikely that there will ever be a need for modelling the US Treasury curve. This is because then the so-called TED spread needs to be measured i.e. the spread between the treasury and AAA curve, which is determined by the swaps. Secondly, the US Treasury is retiring a lot of its debt, so the treasury curve is very illiquid.

[3] Actual/Actual and Actual/360 also occur.

In the above example, the period day count is $\frac{1}{2}$ (as there are two periods in a year) and the elapsed day count is $\frac{115}{182 \cdot 2}$.

This may be the most common convention, and is also known as ISMA Rule 251. Thus, for this calculation let the start date of the period of interest be t_1, the date of interest t, and the end date of the period of interest be t_2. Then on an Actual/Actual basis

$$\text{Accrued period} = \frac{t - t_1}{(t_2 - t_1) \cdot \text{round}(\frac{365}{t_2 - t_1}, 0)} \tag{1.1}$$

$$\text{Period remaining} = \frac{t_2 - t}{(t_2 - t_1) \cdot \text{round}(\frac{365}{t_2 - t_1}, 0)} \tag{1.2}$$

The EURIBOR quotation will be a representative rate for euro deposits based on quotations from a pan-European panel of banks. The euro-LIBOR quotation will be a representative rate for euro deposits based on quotations from a panel of 16 banks in the London market. The conventions for both of these are the 'Euro' conventions above.

1.3 Coupon Bonds

Coupon-bearing bond issuers pay regular fixed interest payments to the holder of the bond on specific dates (these are the coupons) as well as the par or face value at maturity. The bond value at any time t must be the present value of both its face value and its coupons (the coupon rate is pre-fixed e.g. the r153 has a 13% annual coupon, i.e. 6.5% of par value is paid out to the holder every 6 months).

Suppose a bond pays amounts $p_1, p_2, \ldots, p_n$ at times $t_1, t_2, \ldots, t_n$. (p_i does not necessarily mean coupon here, for example, c_n could be the coupon and bullet.) The current bond price V is the sum of the present values of all payments i.e.

$$V = \sum_{i=1}^{n} p_i Z(0, t_i) \tag{1.3}$$

The act of regarding each coupon as a separate zero ensures arbitrage-free pricing. If this were not true, the coupon bond could be synthetically replicated using zero-coupon bonds. This replication is obviously not practically as straightforward as is made out here.

1.4 Yield-to-Maturity

Coupon-bearing bond prices are often quoted in terms of their yield-to-maturity. This is defined as an interest rate per annum that equates the present value of the bond's associated cash flows to the current market price. There exists a terminology confusion here because the yield-to-maturity of a coupon bond does not correspond directly to any value on the zero-coupon yield curve, in particular, it is not the value at the maturity (or duration, to be defined later) on the yield curve, even taking into account possible conversion between different NAC* quoting methods.

The yield-to-maturity is merely a convenient way of expressing the price of a coupon-bearing bond in terms of a single interest rate. In some rough sense, the YTM represents a weighted average of the interest rates along the current zero-coupon yield curve.

If payments are made semi-annually (the most common) then y - the semi-annually compounded yield-to-maturity - is implicitly defined by:

$$V = \sum_{i=1}^{n} \frac{p_i}{(1 + \frac{y}{2})^{2t_i}} \tag{1.4}$$

where there are payments $p_1,\ p_2,\ \ldots,\ p_n$; the payment p_i occurs at time t_i, and time is measured in years.

The market observables are the bond prices V, so the value for y must be calculated numerically. (1.4) cannot be inverted, but Newton's method comes to the rescue. The yield-to-maturity is the holding period return per annum on the coupon bond.

Using the effective yield-to-maturity as the interest rate offered by the bond, implicitly assumes that coupon reinvestment takes place at the semi-annual yield-to-maturity over the bond life. In a world of changing interest rates it is unlikely that this will happen: there is no guarantee that the ytm was an actual market rate at any reinvestment date, even if the yield curve does not change.

Nevertheless, the yield to maturity is a very useful concept: it enables a first order fair comparison of different bonds trading simultaneously. So,

- Typically, the market will trade on price, but market participants will want to know the yield to maturity of the instruments they are trading - they require a yield-given-price algorithm. Here they use Newton's method.

- Less typically (such as in South Africa) the market will trade on yield. We then require a very precise formula for converting from yield to price, in order to determine the cash flows required at the bond exchange. For repos/carrys, the yield-given-price algorithm will be needed anyway.

1.5 Term Structure of Default-Free Interest Rates

The term structure of interest rates is defined as the relationship between the yield-to-maturity on a zero coupon bond and the bond's maturity. If we are going to price derivatives which have been modelled in continuous-time off of the curve, it makes sense to use continuously-compounded rates from the outset.

Building a yield curve from existing data is a difficult task. The liquidity of the market plays a large part in determining whether the exercise can be done. For US Bond market it is only on-the-run bonds (last auctioned) which will give the most accurate indication of where yields are. In South Africa this task is doubly difficult and requires some econometric artistry. South Africa has a sophisticated and liquid swap market. This makes swap rates a better starting point for a yield curve model.

The results of bootstrapping will be near unique in liquid markets, but there may be significant variation in less liquid markets or markets with fewer inputs.

In so-called normal markets, yield curves are upwardly sloping, with longer term interest rates being higher than short term. A yield curve which is downward sloping is called inverted. A yield curve with one or more turning points is called mixed. Constructing a yield curve consists of solving (1.3) for the discount factors $Z(0, t)$ in a piecewise fashion starting with the shortest maturity instruments and progressing to the longer-dated coupon-bearing instruments. A mixed yield curve is shown in Figure 1.1. The discount factors are also shown.

The South African yield curve typically has two to four turning points. It is often stated that such mixed yield curves are signs of market illiquidity or instability. This is not the case. Supply and demand for the instruments that are used to bootstrap the curve may simply imply such shapes. However, many of the models that we see later in this course are driven by one factor - and intuitively this is clearly best suited to a normal or inverted curve (because, essentially, the model dictates that when the curve moves, it more or less moves in parallel). Thus, these models need to be analysed carefully for their suitability in the South African market: see Svoboda [2002].

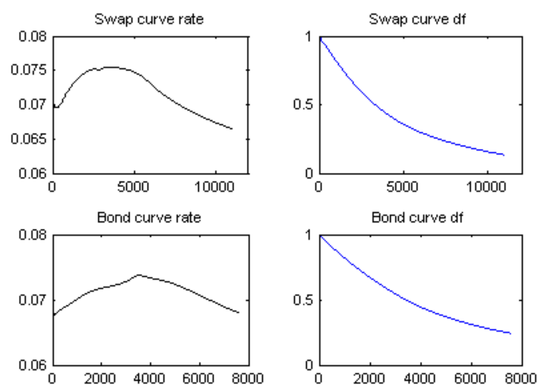


Figure 1.1: Some yield curves and their discount functions

The shape of the graph for $Z(0,t)$ does not reflect the shape of the yield curve in any obvious way. The discount factor curve must - by no arbitrage - be monotonically decreasing whether the yield curve is normal, mixed or inverted. Nevertheless, many bootstrapping and interpolation algorithms for constructing yield curves miss this absolutely fundamental point. See Hagan and West [2006], Hagan and West [2008].

1.6 The par bond curve

Suppose the bond pays ANNUAL coupons of R_n at times $t_1,\ t_2,\ \dots,\ t_n$, with the bullet being at time t_n. The inter-coupon times are α_i, so the i^{th} payment is in fact $R_n\alpha_i$. For it to be a par bond, we must have

$$1 = Z(0,t_n) + \sum_{i=1}^{n} R_n \alpha_i Z(0,t_i) \tag{1.5}$$

We now have a function $t_n \to R_n$ which maps maturity dates to the requisite coupon size. This function is called the par bond curve.

But this should look familiar, and we didn't choose R_n as the notation for the annual coupon size by chance. You see that this is EXACTLY the curve of fair swap rates trading in the market.

Remember that the value of the i^{th} floating payment is

$$\begin{aligned} V^i_{\text{float}} &= Z(t, t_{i-1}) - Z(t, t_i) \\ &= Z(0, t_i)\left[\frac{C(0, t_i)}{C(0, t_{i-1})} - 1\right] \\ &= Z(0, t_i)\alpha_i f^s_i \end{aligned}$$

where f^s_i is the simple forward rate for the period $[t_{i-1}, t_i]$. Thus

$$\begin{aligned} V_{\text{fix}} &= R_n \sum_{i=1}^{n} Z(t, t_i)\alpha_i \\ V_{\text{float}} &= \sum_{i=1}^{n} Z(0, t_i)\alpha_i f^s_i \end{aligned}$$

as as these are equal, we have

$$R_n = \sum_{i=1}^{n} w_i f^s_i \tag{1.6}$$

$$w_i = \frac{Z(t, t_i)\alpha_i}{\sum_{j=1}^{n} Z(t, t_j)\alpha_j} \tag{1.7}$$

Thus the par coupon rates are a weighted average of the simple forward rates along the life of the bond, where the weights are decreasing. On the other hand, we have

$$e^{r\tau} = \prod_{i=1}^{n} e^{f^c_i \alpha_i}$$

and so

$$r = \sum_{i=1}^{n} \frac{\alpha_i}{\tau} f^c_i \tag{1.8}$$

where now f^c_i is the continuous forward rate for the period $[t_{i-1}, t_i]$, and so the zero rates are an almost equally weighted average of the continuous forward rates along the life of the bond.
Allowing for the difference in compounding conventions, we see in general that if the yield curve is normal then the par curve is below the zero curve which is below the forward curve, while if the yield curve is inverted, then the par curve is above the zero curve which is above the forward curve.

1.7 Reminder: Forward Rate Agreements

These are the simplest derivatives: a FRA is an OTC contract to fix the yield interest rate for some period starting in the future. If we can borrow at a known rate at time t to date t_1, and we can borrow from t_1 to t_2 at a rate known and fixed at t, then effectively we can borrow at a known rate at t until t_2. Clearly

$$C(t, t_1)C(t; t_1, t_2) = C(t, t_2) \tag{1.9}$$

is the no arbitrage equation: $C(t; t_1, t_2)$ is the forward capitalisation factor for the period from t_1 to t_2 - it has to be this value at time t with the information available at that time, to ensure no arbitrage.

In a FRA the buyer or borrower (the long party) agrees to pay a fixed yield rate over the forward period and to receive a floating yield rate, namely the 3 month JIBAR rate. At the beginning of the forward period, the product is net settled by discounting the cash flow that should occur at the end of the forward period to the beginning of the forward period at the (then current) JIBAR rate. This feature - which is typical internationally - does not have any effect on the pricing.

Figure 1.2: A long position in a FRA.

In South Africa FRA rates are always quoted for 3 month forward periods eg 3v6, 6v9, ..., 21v24 or even further. The quoted rates are simple rates, so if f is the rate, then

$$1 + f(t_2 - t_1) = C(t; t_1, t_2) = \frac{C(t, t_2)}{C(t, t_1)} \tag{1.10}$$

Thus

$$f = \frac{1}{t_2 - t_1}\left(\frac{C(t, t_2)}{C(t, t_1)} - 1\right) \tag{1.11}$$

1.8 The continuous forward curve

Using continuous rates, the forward rate governing the period from t_1 to t_2, denoted $f(0; t_1, t_2)$ satisfies

$$\exp(-f(0; t_1, t_2)(t_2 - t_1)) = Z(0; t_1, t_2) := \frac{Z(0, t_2)}{Z(0, t_1)}$$

Immediately, we see that forward rates are positive (this is equivalent to the discount function decreasing). We have either of

$$f(0; t_1, t_2) = -\frac{\ln(Z(0, t_2)) - \ln(Z(0, t_1))}{t_2 - t_1} \tag{1.12}$$

$$= \frac{r_2 t_2 - r_1 t_1}{t_2 - t_1} \tag{1.13}$$

Let the instantaneous forward rate for a tenor of t be denoted $f(t)$, that is, $f(t) = \lim_{\epsilon \downarrow 0} f(0; t, t+\epsilon)$, for whichever t this limit exists. Clearly then

$$f(t) = -\frac{d}{dt}\ln(Z(t)) \tag{1.14}$$

$$= \frac{d}{dt} r(t)t \tag{1.15}$$

So $f(t) = r(t) + r'(t)t$, so the forward rates will lie above the yield curve when the yield curve is normal, and below the yield curve when it is inverted. By integrating,[4]

$$r(t)t = \int_0^t f(s)\,ds \tag{1.16}$$

$$= r(t_{i-1})t_{i-1} + \int_{t_{i-1}}^t f(s)\,ds \tag{1.17}$$

$$Z(t) = \exp\left(-\int_0^t f(s)\,ds\right) \tag{1.18}$$

Also

$$f_i^d := \frac{r_i t_i - r_{i-1}t_{i-1}}{t_i - t_{i-1}} = \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(s)\,ds \tag{1.19}$$

which shows that the average of the instantaneous forward rate over any interval $[t_{i-1}, t_i]$ is equal to the discrete forward rate for that interval.

1.9 The raw interpolation method for yield curve construction

This method corresponds to piecewise constant forward curves. This method is very stable, is trivial to implement, and is usually the starting point for developing models of the yield curve. One can often find mistakes in fancier methods by comparing the raw method with the more sophisticated method.

By definition, raw interpolation is the method which has constant instantaneous forward rates on every interval $t_{i-1} < t < t_i$. From (1.19) we see that that constant must be the discrete forward rate for the interval, so $f(t) = \frac{r_i t_i - r_{i-1}t_{i-1}}{t_i - t_{i-1}}$ for $t_{i-1} < t < t_i$. Then from (1.17) we have that

$$r(t)t = r_{i-1}t_{i-1} + (t - t_{i-1})\frac{r_i t_i - r_{i-1}t_{i-1}}{t_i - t_{i-1}}$$

By writing the above expression with a common denominator of $t_i - t_{i-1}$, and simplifying, we get that the interpolation formula on that interval is

$$r(t)t = \frac{t - t_{i-1}}{t_i - t_{i-1}} r_i t_i + \frac{t_i - t}{t_i - t_{i-1}} r_{i-1}t_{i-1} \tag{1.20}$$

which explains yet another choice of name for this method: 'linear rt'; the method is linear interpolation on the points $r_i t_i$. Since $\pm r_i t_i$ is the logarithm of the capitalisation/discount factors, we see that calling this method 'linear on the log of capitalisation factors' or 'linear on the log of discount factors' is also merited.

This raw method is very attractive because with no effort whatsoever we have guaranteed that all instantaneous forwards are positive, because every instantaneous forward is equal to the discrete forward for the 'parent' interval. This is an achievement not to be sneezed at. It is only at the points $t_1, t_2, \ldots, t_n$ that the instantaneous forward is undefined, moreover, the function jumps at that point.

[4] We have $r(s)s + C = \int f(s)\,ds$, so $r(t)t = [r(s)s]_0^t = \int_0^t f(s)\,ds$.

1.10 Traditional Measures of Interest Rate Risk

Traditionally interest rate (bond) traders used two measures of interest rate risk.

1.10.1 (Macauley) Duration and Modified Duration

Bonds might alternatively be priced not off of the yield curve but using a NACS yield to maturity. What happens then? Suppose our bond is priced NACn, then

$$V(y) = \sum_{i=1}^{n} \frac{p_i}{(1+\frac{y}{n})^{nt_i}}$$

This time

$$w_t = \frac{\frac{p_i}{\left(1+\frac{y}{n}\right)^{nt_i}}}{V}$$

and $D = \sum_{i=1}^{n} t_i w_i$ is as before. But now something changes. This time

$$\frac{dV}{dy} = -\frac{1}{1+\frac{y}{n}} \sum_{i=1}^{n} \frac{t_i p_i}{(1+\frac{y}{n})^{nt_i}}$$

so it is not true that $\frac{dV}{dy} = -DV$. So what we do is define a new quantity, called modified duration D_m, by:

$$D_m V = \frac{1}{1+\frac{y}{n}} \sum_{i=1}^{n} \frac{t_i p_i}{(1+\frac{y}{n})^{nt_i}} \tag{1.21}$$

so

$$\frac{dV}{dy} = -D_m V \tag{1.22}$$

We also have the definition of the dollar duration

$$D_{\$} = D_m V \tag{1.23}$$

Also, note that

$$D_m = \frac{D}{1+\frac{y}{n}} \tag{1.24}$$

As $n \to \infty$, the correction between duration and modified duration disappears, and they become the same thing for NACC rates.

1.10.2 Convexity

Of course there is a non-linear relationship between the bond price and the yield-to-maturity. For small shifts in yield, the first order duration calculation is a good measure of the sensitivity of the bond price. If Δy is large, the approximation is no longer accurate.

Using a Taylor expansion to second-order we can define convexity.

$$\begin{aligned} \Delta V &= V(y+\Delta y) - V(y) \\ &= \frac{dV}{dy}\Delta y + \tfrac{1}{2}\frac{d^2 V}{dy^2}(\Delta y)^2 + O\left(\Delta y^3\right) \end{aligned}$$

and now convexity C is defined as

$$CV = \frac{d^2V}{dy^2} \tag{1.25}$$

so we have

$$\frac{\Delta V}{V} = -D_m \Delta y + \tfrac{1}{2} C \Delta y^2 + O\left(\Delta y^3\right) \tag{1.26}$$

1.10.3 Problems with these measures

Because the yield-to-maturity is, in some sense, a measure of the average rate of interest represented by the yield curve with respect to the bond in question, information about more subtle yield curve shifts is lost. It is possible for the change in the yield-to-maturity to mask the underlying zero coupon curve fluctuations.

By using duration measurements as interest rate risk measurement across different bonds, it is assumed that each bond experiences the same yield-to-maturity shift as the yield curve moves. This only occurs if the zero coupon curve moves in a parallel fashion i.e. the whole curve moves up or down. We have already discussed how, in general, this is not the case. Furthermore, for large yield shifts, the convexity correction will not capture the price shift, so by using this measure we are trapped in a small-movement, parallel-shift regime. This makes for naïve hedging.

1.10.4 pv01

The pv01 of an instrument is how much its price changes when the yield changes by 1 basis point. We also discard the minus sign. Since a basis point is 0.0001, a general result is that

$$\text{pv01} = -0.0001 \frac{dV}{dy} \tag{1.27}$$

If we are considering a bond, then using (1.23) we have

$$\text{pv01} = 0.0001 D_{\$} \tag{1.28}$$

Alternatively, if we have a yield curve, then it is the change in price when the curve moves in parallel by 1 basis point. Then

$$\text{pv01} = -0.0001 \frac{dV}{d\bar{r}} \tag{1.29}$$

Duration calculations can give silly results, and usually the task at hand can best be handled with pv01 instead.

For example, what is the duration of a swap? We can calculate $\frac{\partial V}{\partial y}$, but to get the duration we must (negate and) divide by V, which could be close to (or exactly!) 0, and hence we have numerical instability. Rather we should stick to the pv01 calculation and be done with it.

Usually the duration calculation should be done at the portfolio level, and then a swap should be split up as short a fixed coupon bond and long a floating rate note. The fixed coupon bond duration is found as normal and the floating rate note has a duration which is at most the time to the next reset.

If one wants to have a robust definition of the duration of a swap on a stand alone basis, one should say it is the duration of the floating rate note minus the duration of the fixed coupon bond. The pv01 of a just starting swap is equal to the duration of the short fixed coupon bond, because the duration of a just starting floating rate note is 0.

1.11 Exercises

1. The current discount rate, assuming a 360 day year, on a 90-day bill is 3.5%. The face value is $1 million.
   (a) What is the price of the bill?
   (b) If the discount rate increases by one basis point to 3.51%, what is the *change* in the price of the bill?
   (c) If the discount rate decreases by one basis point to 3.49% percent, what is the *change* in the price of the bill?
   (d) Is the price function symmetric?
2. The continuously-compounded yield on a deposit which pays one million ZAR in 363 days' time is 8.55%.
   (a) What is the current value of the deposit?
   (b) What is the yield expressed as a simple interest rate, assuming a 365 day year?
3. The current discount rate on a 91-day bill is 3.68 percent, assuming a 360 day year. What is the simple interest rate, assuming a 365 day year?
4. Use excel to find the ytm of a NACS bond. So, the inputs are a bunch of cash flows, their dates of payment, today's date, and the total cash price; the output needs to be the ytm. Make sure that not only newly issued bonds can be catered for. Use an actual/365 day count. Write a macro which, each time it is played, calculates the next estimate of the ytm, based on the previous one, using the Newton-Bailey method.
5. Write a vba function that, given
   (a) period start date;
   (b) current date;
   (c) period end date;
   (d) day count convention (abbreviated name).

   will calculate the portion of the year already elapsed.
6. Show that if a newly issued bond prices at par then the ytm is equal to the coupon (with the frequency of the ytm being the frequency of the coupon).
7. Create a spreadsheet which, given the yield curve as input, calculates the par bond curve and the forward curve. Graph and label all three curves.

8. Create a spreadsheet that given a set of semi-annual bond cash flows with their dates, the ytm of the bond, and today's date, can calculate the duration, modified duration, and convexity of the bond. The bond is not necessarily newly issued. Use an actual/36 day count. Fill out the cash flows for the r153 and compare the answers you get with the answers you get from your SAFM bond calculator.

9. A similar exercise: create a spreadsheet that given the yield curve values, a set of semi-annual bond cash flows with their dates, and today's date, can calculate the duration and convexity of the bond. Do not calculate a ytm.

Chapter 2

Discrete-time Interest Rate Models

2.1 Introduction

The problem of interest rate derivatives can be approached broadly in one of two ways. We can either model the interest rate process or the bond price process. If we are modelling the interest rate process, then we must decide which interest rate(s) it is that we will model. In either case, there are consistency conditions that must hold because of the structure of the yield curve: in particular, implied forward rates should be positive.

In a discrete-time framework, we can use a similar model to that used in the binomial model of stock price movements to model the bond prices and/or the interest rates. Modelling the price of the bond requires us to ensure that the process is arbitrage-free, consistent with the initial term-structure and obeys the boundary condition - that the price of a maturing bond is par, plus possibly the final coupon.

In this chapter we will model the evolution of the short-rate (defined as the continuously-compounded rate over Δt, the discrete time period for the lattice). The yield curve evolution is then governed by one underlying factor. Substantial literature is devoted to multifactor models, where either more than one source of uncertainty is modelled (usually principle components or their proxies), or the evolution of the entire forward curve is modelled.

One-factor models are simple and tractable but they are not very flexible. In particular, they will be hampered by their inability within the model for certain yield curve shapes to occur. In the most simple models interest rates can become negative.

2.2 The basic lattice construction

Consider the following NACC term structure, with $\Delta t = 1$:

i	1	2	3	4	5	6	7	8
r_i	6.1982%	6.4030%	6.8721%	7.0193%	7.2000%	6.9000%	6.9000%	7.0000%
σ_i	1.7000%	1.5000%	1.1000%	1.0000%	1.0000%	1.0000%	1.1000%	

The risk free rate r_i is the NACC rate for the period $[0, t_i]$.
The volatilities are not the bond price volatilities. They are also not the volatilities of the risk free rates that are from my yield curve bootstrap. They are the forward volatilities of the spot rate (over Δt) over the time period $[t_i, t_{i+1}]$, given the spot rates at time $t = 0$. In other words, they are the volatilities for the short rate for the period $[t_i, t_{i+1}]$ that will be observed at time t_i.
Of course, there is no σ_0: the spot rate for the period from time $0 = t_0$ to time $t_1 = \Delta t$ is known. Similarly, given that the horizon in the above example is t_8, the volatility in the period $[t_8, t_9]$ is irrelevant.
We use binomial trees for the Δt rate, the general notational convention of a tree will be as in Figure 2.1.

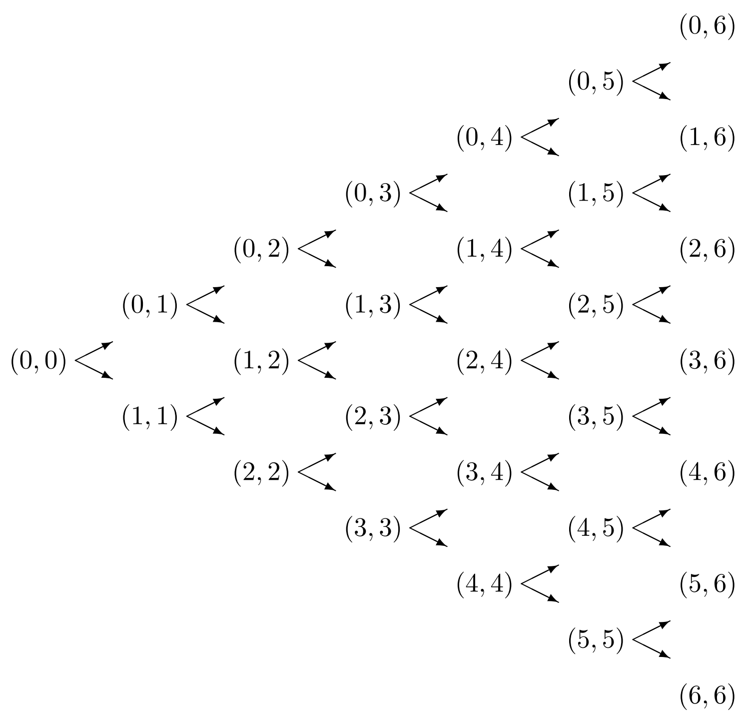


Figure 2.1: General node notation for binomial trees

Using the binomial assumption the spot rate in Δt time takes one of two values: $r^t(0, 1)$ or $r^t(1, 1)$, corresponding are the discount factors $Z^t(0, 1)$ and $Z^t(1, 1)$.[1]

2.3 Normal Distribution (Ho-Lee model)

The values we use here will be seen to be consistent with assuming that the spot rate process is normally distributed: this is the Ho and Lee model Ho and Lee [1986].

[1]The superscript t means 'tree', in order to avoid confusion with the original discount factors from the original yield curve. $Z^t(i, j)$ is the discount factor for the following Δt period given that we have evolved to node (i, j). So $Z^t(0, 0) = Z(0, t_1)$ (!!).

Using martingale pricing theory on bond prices (analogously to stock prices) we know that there exists a unique probability measure $Q = \{\pi\}$ such that the bond price normalised by the money-market account follows a martingale. Under the martingale measure, the current forward bond price is the expected price. Then

$$\begin{aligned} Z(0,2) &= \mathbb{E}_0^{\mathbb{Q}}[Z(1,2)Z(0,1)] \\ &= Z(0,1)\mathbb{E}_0^{\mathbb{Q}}[0]\, Z(1,2) \\ &= Z(0,1)[\pi Z^t(0,1) + (1-\pi)Z^t(1,1)] \end{aligned} \tag{2.1}$$

where π denotes the probability of an up move and $1-\pi$ the probability of a down move.
Now the interest rates $r^t(1,1)$ and $r^t(0,1)$ must match the volatility term structure. Note that in general, if a variable r can take on two values, a and b, with $a > b$, and with probabilities π and $1-\pi$ respectively, then the variance of r is given by

$$\begin{aligned} \text{variance}(r) &= \mathbb{E}^{\mathbb{Q}}\left[r^2\right] - (\mathbb{E}^{\mathbb{Q}}[r])^2 \\ &= \pi a^2 + (1-\pi)b^2 - (\pi a + (1-\pi)b)^2 \\ &= (a-b)^2\pi(1-\pi) \end{aligned} \tag{2.2}$$

So, it is very good idea to assume that $\pi = \frac{1}{2}$. (As usual, the model has three free parameters and is matching two characteristic equations, the mean and variance. So there is one degree of freedom, which we now use up.) Then

$$\begin{aligned} Z(0,2) &= Z(0,1)\tfrac{1}{2}[e^{-a} + e^{-b}] \\ \text{stdev}(r) &= \tfrac{1}{2}(a-b) \end{aligned}$$

which is two equations in two unknowns, and solves easily:[2] $r^t(1,1) = 8.3223\%$, $r^t(1,0) = 4.9223\%$.
Now extend the lattice from one to two years, and assume that the volatility is time-dependent but not state-dependent (i.e. volatility varies from left to right and not from top to bottom). Furthermore, we will require that the tree recombines. This information is enough to calibrate the tree in closed inductive form.
But first, in order to explore a bit further, some information from heaven: guess that the values for the short-rate at time $t = 2$ are $r^t(2,2) = 10.8523\%$, $r^t(1,2) = 7.8583\%$ and $r^t(0,2) = 4.8583\%$. What do we need to do to check that this heavenly information is correct? First, and easily, the volatility structure:

$$\begin{aligned} \tfrac{1}{2}(r^t(2,2) - r^t(1,2)) &= \tfrac{1}{2}(0.108583 - 0.078583) = 0.015 \\ \tfrac{1}{2}(r^t(1,2) - r^t(0,2)) &= \tfrac{1}{2}(0.078583 - 0.048583) = 0.015 \end{aligned}$$

Now,

$$\begin{aligned} Z(0,3) &= Z(0,1)\mathbb{E}_0^{\mathbb{Q}}[Z(1,3)] \\ &= Z(0,1)\mathbb{E}_0^{\mathbb{Q}}[Z(1,2)Z(2,3)] \\ &= Z(0,1)[\tfrac{1}{2}Z^t(1,1)\mathbb{E}_0^{\mathbb{Q}}[Z(2,3)|(1,1)] + \tfrac{1}{2}Z^t(0,1)\mathbb{E}_0^{\mathbb{Q}}[Z(2,3)|(0,1)]] \\ &= Z(0,1)[\tfrac{1}{2}Z^t(1,1)\tfrac{1}{2}[Z^t(2,2) + Z^t(1,2)] + \tfrac{1}{2}Z^t(0,1)\tfrac{1}{2}[Z^t(1,2) + Z^t(0,2)] \end{aligned}$$

[2] It doesn't solve easily because it is two equations in two unknowns; that fact means that there should be a solution. The ease of the solution comes from convenient facts of the exponential.

Exercise 2.3.1. We know from the previous steps the values of $Z(0,1)$, $Z^t(1,1)$ and $Z^t(0,1)$. Check that the heavenly values work in the above equation.

This makes the model parameters consistent with all the market information provided, and hence the model is arbitrage free.

2.4 Formalising the Lattice Construction

We will have the binomial lattice of discount factors $Z^t(i,j)$, which are the discount factors over the following period if at time j we are in state i. Typically the periods are three months apart. Here i is a state (vertical) index and j is a time (horizontal) index.
Let $\sigma(j)$ be the annualised volatility of the Δt month forward rates for the period $[t_j, t_{j+1}]$. In the absence of implied information, the volatilities can only be calibrated via an historical analysis of the yield curve. This volatility will be calibrated under the Ho-Lee type assumption of normality of interest rates i.e. one takes differences of rates rather than log differences of rates in the calculation estimating standard deviations. Nevertheless, the square root of time rule for volatilities still applies, because we are assuming a Brownian evolution of the rate.
In the Ho-Lee model, there are equal probabilities for evolving to the two subsequent nodes in the tree, and furthermore by a normality assumption the difference in adjacent interest rates at time j for the period from then to time $j+1$ is independent of the state i, so

$$r^t(i+1,j) = r^t(i,j) + 2\sigma(j)\sqrt{\Delta t} \tag{2.3}$$

Hence

$$\begin{aligned} Z^t(i+1,j) &= Z^t(i,j)E(j) \ (1 \le i \le j-1) && (2.4) \\ E(j) &:= \exp\left[-2\sigma(j)\Delta t^{3/2}\right] && (2.5) \end{aligned}$$

To proceed, define new variables $\lambda(i,j)$. This variable is called the Arrow-Debreu price, and is the price at time 0 of a security that pays off exactly one if we pass through the node (i,j), the payoff occurring at the moment of passing through. Thus

$$\begin{aligned} \lambda(0,0) &= 1 && (2.6) \\ \lambda(0,j) &= \tfrac{1}{2}\lambda(0,j-1)Z^t(0,j-1) && (2.7) \\ \lambda(i,j) &= \tfrac{1}{2}\lambda(i-1,j-1)Z^t(i-1,j-1) + \tfrac{1}{2}\lambda(i,j-1)Z^t(i,j-1) \ (0 < i < j) && (2.8) \\ \lambda(j,j) &= \tfrac{1}{2}\lambda(j-1,j-1)Z^t(j-1,j-1) && (2.9) \end{aligned}$$

Now, by no arbitrage we have

$$Z(0,(j+1)\Delta t) = \sum_{i=0}^{j} \lambda(i,j)Z^t(i,j) \tag{2.10}$$

and now recalling (2.4) we have

$$Z(0,(j+1)\Delta t) = \sum_{i=0}^{j} \lambda(i,j)Z^t(i,j) = Z^t(0,j)\sum_{i=0}^{j} \lambda(i,j)E(j)^i$$

$\lambda(i,j)$	0	1	2	3	4	5	6	7	8
0	1.0000	0.4699	0.2237	0.1065	0.0511	0.0245	0.0122	0.0060	0.0030
1		0.4699	0.4399	0.3099	0.1963	0.1170	0.0692	0.0396	0.0223
2			0.2162	0.3003	0.2830	0.2230	0.1636	0.1113	0.0724
3				0.0970	0.1813	0.2126	0.2061	0.1737	0.1342
4					0.0435	0.1013	0.1460	0.1627	0.1553
5						0.0193	0.0552	0.0914	0.1151
6							0.0087	0.0285	0.0533
7								0.0038	0.0141
8									0.0016
$Z^t(i,j)$	0	1	2	3	4	5	6	7	
0	0.9399	0.9520	0.9526	0.9586	0.9606	0.9946	0.9891	0.9971	
1		0.9201	0.9244	0.9377	0.9416	0.9749	0.9695	0.9754	
2			0.8971	0.9173	0.9229	0.9556	0.9503	0.9541	
3				0.8974	0.9046	0.9366	0.9315	0.9334	
4					0.8867	0.9181	0.9130	0.9131	
5						0.8999	0.8949	0.8932	
6							0.8772	0.8738	
7								0.8547	
$r^t(i,j)$	0	1	2	3	4	5	6	7	
0	6.198%	4.922%	4.858%	4.231%	4.023%	0.545%	1.100%	0.295%	
1		8.322%	7.858%	6.431%	6.023%	2.545%	3.100%	2.495%	
2			10.858%	8.631%	8.023%	4.545%	5.100%	4.695%	
3				10.831%	10.023%	6.545%	7.100%	6.895%	
4					12.023%	8.545%	9.100%	9.095%	
5						10.545%	11.100%	11.295%	
6							13.100%	13.495%	
7								15.695%	

Figure 2.2: The Ho-Lee trees associated with the given data

$$Z^t(0,j) = \frac{Z(0,(j+1)\Delta t)}{\sum_{i=0}^{j} \lambda(i,j)E(j)^i} \tag{2.11}$$

Thus our recursion is as follows, at time step j:

(a) Calculate $\lambda(i,j)$ for $0 \le i \le j$ from (2.6), (2.7), (2.8) and (2.9).

(b) Calculate $Z^t(0,j)$ from (2.11).

(c) Calculate $Z^t(i,j)$ for $1 \le i \le j$ from (2.4).

Exercise 2.4.1. Verify Figure 2.2 for the given data.

2.5 Lognormal Distribution (Black-Derman-Toy model)

Discrete time models of interest rates where the forward rates are lognormally distributed are usually taken to be some variation of Black et al. [1990]. The problem with the normality assumption in the previous section is that it is possible for interest rates to become negative (even if a mean reversion factor is introduced). Clearly, a lognormal assumption precludes this.

Assume now that the logarithm of the spot rate is normally distributed, and the volatility parameter pertains to the logarithms of the interest rates rather than the interest rates themselves. So now

$$\begin{aligned} \ln r^t(i+1,j) &= \ln r^t(i,j) + 2\sigma(j)\sqrt{\Delta t} \\ r^t(i+1,j) &= r^t(i,j)\exp(2\sigma(j)\sqrt{\Delta t}) \end{aligned} \tag{2.12}$$

Hence

$$Z^t(i+1,j) \quad = \quad Z^t(i,j)^{E(j)} \ (1 \le i \le j-1) \tag{2.13}$$

$$E(j) \quad := \quad \exp\left[2\sigma(j)\sqrt{\Delta t}\right] \tag{2.14}$$

which are calculated in advance. To proceed, again define new variables $\lambda(i,j)$ exactly as before: (2.6), (2.7), (2.8), (2.9) and (2.10) are unchanged. Thus

$$Z(0,(j+1)\Delta t) = \sum_{i=0}^{j} \lambda(i,j) Z^t(i,j) = \sum_{i=0}^{j} \lambda(i,j) Z^t(0,j)^{E(j)^i}$$

which does not have a closed form solution. Hence, we need to solve this for $Z^t(0,j)$ numerically. Let $x = Z^t(0,j)$. Using Newton's method, we solve

$$\begin{aligned} x_1 &= Z^t(0,j-1) \\ x_{n+1} &= x_n - \frac{\sum_{i=0}^{j} \lambda(i,j) x_n^{E(j)^i} - Z(0,(j+1)\Delta t)}{\sum_{i=0}^{j} \lambda(i,j) E(j)^i x_n^{E(j)^i - 1}} \end{aligned}$$

Convergence here is extremely rapid: to double precision in 3 or 4 iterations. The Newton function above is near linear (in a very wide range). Moreover, the iteration can be performed simultaneously across all j.

Note that the term structure of volatilities in this lognormal model will reflect higher values than those under the normality assumption. We see this via the following:

$$\begin{aligned} \ln[r + \Delta r] - \ln[r] &= \ln[r(1 + \tfrac{\Delta r}{r})] - \ln[r] \\ &= \ln[1 + \tfrac{\Delta r}{r}] \\ &\simeq {}^{3}\, \tfrac{\Delta r}{r} \end{aligned}$$

for small Δ. This implies that,

$$\sigma_{\text{lognormal}} = {}^{4}\, \frac{\sigma_{\text{normal}}}{r} \tag{2.15}$$

[3] Recall that the Taylor series of $\ln(1+x)$ is $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$.

[4] So, the classic rule of thumb in South Africa: to get from market volatilities to ballpark Ho-Lee volatilities: knock off a decimal place. Quoted volatilities will be lognormal because use of Black's model is the default: see Chapter 3.

Exercise 2.5.1. Assume the following initial data, with $\Delta t = \frac{1}{4}$:

i	1	2	3	4	5	6	7	8
r_i	6.1982%	6.4030%	6.8721%	7.0193%	7.1000%	7.2021%	7.3120%	7.3000%
σ_i	20.0000%	18.0000%	17.0000%	17.0000%	17.0000%	17.0000%	17.0000%	

Check Figure 2.1.

$\lambda(i,j)$	0	1	2	3	4	5	6	7	8
0	1.0000	0.4923	0.2425	0.1193	0.0588	0.0290	0.0143	0.0071	0.0035
1		0.4923	0.4842	0.3568	0.2342	0.1443	0.0854	0.0492	0.0278
2			0.2417	0.3556	0.3496	0.2870	0.2121	0.1465	0.0966
3				0.1181	0.2319	0.2851	0.2806	0.2419	0.1912
4					0.0577	0.1416	0.2087	0.2396	0.2365
5						0.0281	0.0827	0.1422	0.1870
6							0.0136	0.0468	0.0923
7								0.0066	0.0260
8									0.0032
$Z^t(i,j)$	0	1	2	3	4	5	6	7	
0	0.9846	0.9852	0.9839	0.9858	0.9871	0.9877	0.9883	0.9903	
1		0.9820	0.9808	0.9832	0.9847	0.9854	0.9862	0.9885	
2			0.9771	0.9801	0.9819	0.9827	0.9836	0.9864	
3				0.9765	0.9785	0.9796	0.9806	0.9839	
4					0.9746	0.9758	0.9771	0.9810	
5						0.9714	0.9729	0.9775	
6							0.9680	0.9734	
7								0.9685	
$r^t(i,j)$	0	1	2	3	4	5	6	7	
0	6.198%	5.950%	6.473%	5.723%	5.213%	4.961%	4.696%	3.894%	
1		7.267%	7.750%	6.783%	6.179%	5.880%	5.566%	4.616%	
2			9.278%	8.041%	7.325%	6.970%	6.598%	5.471%	
3				9.530%	8.682%	8.261%	7.820%	6.485%	
4					10.291%	9.792%	9.270%	7.687%	
5						11.606%	10.987%	9.111%	
6							13.023%	10.799%	
7								12.800%	

Table 2.1: The Black-Derman-Toy trees associated with the given data

2.6 Options on Zero-Coupon Bonds

The procedure that we follow is the usual martingale pricing approach. We generate an (arbitrage-free) tree of discount factors and then use risk neutral valuation. Using this basis we can also price complicated interest rate derivatives. In general, the option will have maturity T and will be written on an instrument with maturity T_1 where $(T_1 > T)$.

As a simple example, consider such an option on a zero coupon bill. By arbitrage considerations, the value of the bill at maturity of the option T will be $Z(T, T_1)$. The value of such a bill today is $Z(0; T, T_1)$ ie. the forward discount value. But to value the option, we need to consider the volatility. Then the European call and put boundary conditions are:

$$c[Z(T,T_1),0;K] = \begin{cases} Z(T,T_1) - K & \text{for} \quad Z(T,T_1) > K \\ 0 & \text{for} \quad Z(T,T_1) \leq K \end{cases} \tag{2.16}$$

$$p[Z(T,T_1),0;K] = \begin{cases} 0 & \text{for} \quad Z(T,T_1) \geq K \\ K - Z(T,T_1) & \text{for} \quad Z(T,T_1) < K \end{cases} \tag{2.17}$$

respectively.

As an example we will consider a 18 month option on a six month zero-coupon bond i.e. after 18 months we decide whether or not to buy/sell a zero-coupon bond at the strike price; the zero coupon bond pays 1 after 2 years.

We use the Black-Derman-Toy model with the previous data. Given the tree of prices, there is very little to do. All we need to realise is that at time t_6 the bond has a value of

$$V(i,6) = Z^t(i,6)\tfrac{1}{2}\left[Z^t(i,7) + Z^t(i+1,7)\right] \tag{2.18}$$

Thus, the payoff of the option is

$$V^t(i,6) = \max(\eta(V(i,6) - K, 0))$$

The value of the option now is

$$V(0) = \sum_{i=0}^{6} \lambda(i,6) V^t(i,6)$$

Alternatively we can induct backwards through the tree as usual. There are benefits to the latter approach as it will enable us to derive hedge ratios.

Verify that, for a call option on a six month zero coupon bond, strike 0.95, we get the tree of option values

0	1	2	3	4	5	6
0.0117	0.0146	0.0175	0.0203	0.0230	0.0255	0.0279
	0.0091	0.0121	0.0152	0.0183	0.0211	0.0238
		0.0065	0.0094	0.0127	0.0160	0.0191
			0.0039	0.0065	0.0099	0.0134
				0.0016	0.0033	0.0068
					0.0000	0.0000
						0.0000

and so a price of 0.0117.

Put-Call parity for bond options is analogous to that of European options written on non-dividend paying stocks. Consider European put and call options written on zero-coupon bonds with $par = 1$. The option maturity is T and the bond maturity is T_1, where $T_1 > T$. Then

$$p[Z(t,T_1), T-t; K] + Z(t,T_1) = c[Z(t,T_1), T-t; K] + KZ(t,T) \tag{2.19}$$

2.7 Forwards on Zero-Coupon Bonds

The forward value of the bond is

$$V(0) = \sum_{i=0}^{6} p_i V(i,6)$$

where p_i is the probability of arriving at the node i. These probabilities may be found using Pascal's triangle, or directly via combinatorial arguments; $p_i = 2^{-6}\binom{6}{i}$. Hence the forward value is

$$V(0) = 2^{-6} \sum_{i=0}^{6} \binom{6}{i} V^t(i,6)$$

2.8 Hedging Options

Replicating portfolios for options can be constructed using bonds or forwards. Consider the BDT example as previously. The hedging portfolio must contain two instruments (because of the binomial tree structure). We can choose any two of the zero coupon bonds that can be found in the market. Let us choose bonds maturing at time t_1 and t_2; we seeking hedge quanta n_1 and n_2.

$$\begin{aligned} V(0) &= Z(0,t_1)n_1 + Z(0,t_2)n_2 \\ V^t(0,1) &= n_1 + Z^t(0,1)n_2 \\ V^t(1,1) &= n_1 + Z^t(1,1)n_2 \end{aligned}$$

in other words

$$\begin{aligned} 0.0117 &= 0.9846n_1 + 0.9685n_2 \\ 0.0146 &= n_1 + 0.9852n_2 \\ 0.0091 &= n_1 + 0.9820n_2 \end{aligned}$$

Although this appears to be an over-specified system, it isn't really, as the first equation follows from the others, so we discard it. Thus we have a 2×2 system:

$$\begin{bmatrix} 1 & 0.9852 \\ 1 & 0.9820 \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} 0.0146 \\ 0.0091 \end{bmatrix}$$

which solves as $n_1 = -1.64$, $n_2 = 1.6793$ by using Cramer's rule, say.

What can you say about the effectiveness of this hedge? How can it be modified to be more robust?

Hedging can also be done with 2-period forwards. The tree of forwards prices is:

0	1	2	3	4	5	6
0.9627	0.9658	0.9686	0.9712	0.9736	0.9758	0.9779
	0.9596	0.9629	0.9660	0.9688	0.9714	0.9738
		0.9562	0.9598	0.9632	0.9662	0.9691
			0.9526	0.9565	0.9601	0.9634
				0.9487	0.9529	0.9568
					0.9444	0.9490
						0.9398

Here, the last column is just the values of (2.18). We then induct to the left: the value in cell (i, j) is just the average of cells $(i, j+1)$ and $(i+1, j+1)$. As rates evolve, so the forward price 'converges' to the actual price after 18 months. Define the delta of the option with respect to the forwards contract in the usual fashion:

$$\Delta = \frac{V^t(0,1) - V^t(1,1)}{V_f^t(0,1) - V_f^t(1,1)} = \frac{0.0146 - 0.0091}{0.9658 - 0.9596} = 0.8763$$

Then replicate the option with a forward position and a cash account, with quanta m_1 and m_2 respectively:

$$\begin{aligned} V(0) &= m_1\, 0 + m_2\, 1 \\ V^t(0,1) &= m_1[V_f^t(0,1) - V_f^t(0,0)] + m_2 Z(0,1)^{-1} \\ V^t(1,1) &= m_1[V_f^t(1,1) - V_f^t(0,0)] + m_2 Z(0,1)^{-1} \end{aligned}$$

in other words

$$\begin{aligned} 0.0117 &= m_2 \\ 0.0146 &= [0.9658 - 0.9627]m_1 + 1.0156 m_2 \\ 0.0091 &= [0.9596 - 0.9627]m_1 + 1.0156 m_2 \end{aligned}$$

which implies that $m_1 = 0.8763$, $m_2 = 0.0117$. As usual, the hedge ratio is dynamic and we need to re-balance the hedge portfolio at $t = 1$, depending on up/down movement from $t = 0$.

2.9 Exercises

1. Consider the following quarterly data, satisfying the conditions of the Black-Derman-Toy model:

	1	2	3	4	5	6	7	8
zero	6.00%	6.50%	6.30%	7.00%	7.00%	7.00%	7.00%	7.00%
vol	20.00%	18.00%	17.00%	16.00%	15.00%	15.00%	14.00%	

   (1) Create arrays in vba that will store these rates.

   (2) Type these rates into excel, and write vba code to read them (using a loop) into the two arrays.

   (3) Create in the same way as above arrays ArrowDebreu(0 to 8,0 to 8), Bt(0 to 7,0 to 7), and ratet(0 to 7, 0 to 7).

   (4) Create the time index j and the state index i. By looping through time, populate (the upper triangle) of all of these matrices. Here you use Newton iteration.

   Remark: all your dims should be together, at the start of your code. It is bad form to dim things only as you need them. However, the redims can be in the heart of the matter. (A more sophisticated approach will have the number of data points as a variable and the redims can involve this variable. So, by necessity, the redim can only occur after some calculation work has been done.)

(5) Use loops to again 'print' the output values in excel.

(6) Price the options and forwards seen in class in vba.

Chapter 3

Black's Model

We consider the Black Model for futures/forwards which is the market standard for quoting prices (via implied volatilities). Black [1976] considered the problem of writing options on commodity futures and this was the first "natural" extension of the Black-Scholes model. This model also is used to price options on interest rates and interest rate sensitive instruments such as bonds. Since the Black-Scholes analysis assumes constant (or deterministic) interest rates, and so forward interest rates are realised, it is difficult initially to see how this model applies to interest rate dependent derivatives.
However, if f is a forward interest rate, it can be shown that it is consistent to assume that

- The discounting process can be taken to be the existing yield curve.
- The forward rates are stochastic and log-normally distributed.

The forward rates will be log-normally distributed in what is called the T-forward measure, where T is the pay date of the option. This model is consistent is within the domain of the LIBOR market model. We can proceed to use Black's model without knowing any of the theory of the LMM; however, Black's model cannot safely be used to value more complicated products where the payoff depends on observations at multiple dates.

3.1 European Bond Options

The clean (quoted) price for a bond is related to the all-in (dirty, cash) price via:

$$\mathbb{A} = \mathbb{C} + \mathbb{I}_A(t) \tag{3.1}$$

where the accrued interest $\mathbb{I}_A(t)$ is the accrued interest as of date t, and is non-zero between coupon dates. The forward price is a carried all-in price, not a clean price. The option strike price $\mathbb{K}$ might be a clean or all-in strike; usually it is clean. If so, we change it to a all-in price by replacing $\mathbb{K}$ with $\mathbb{K}_{\mathbb{A}} = \mathbb{K} + \mathbb{I}_{\mathbb{A}}(T)$.

Applying Black's model to the price of the bond, the value of the bond option per unit of nominal is [Hull, 2005, §26.2]

$$V_\eta = \eta Z(0,T)[\mathbb{F}_\mathbb{A} N(\eta d_1) - \mathbb{K}_\mathbb{A} N(\eta d_2)] \tag{3.2}$$

$$d_{1,2} = \frac{\ln \frac{\mathbb{F}_\mathbb{A}}{\mathbb{K}_\mathbb{A}} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{3.3}$$

where $\eta = 1$ stands for a call, $\eta = -1$ for a put. Here σ is the volatility measure of the fair forward all in price, and τ as usual is the term of the option in years with the relevant day-count convention applied.

Example 3.1.1. Consider a $10m$ European call option on a $1,000,000$ bond with 9.75 years to maturity. Suppose the coupon is 10% NACS. The clean price is 935,000 and the clean strike price is 1,000,000. We have the following yield curve information:

Term	Rate
3m	9%
9m	9.5%
10m	10%

The 10 month volatility on the bond price is 9%.

Firstly, $\mathbb{K}_\mathbb{C} = 1,000,000$ so $\mathbb{K}_\mathbb{A} = \mathbb{K}_\mathbb{C} + \mathbb{I}_\mathbb{A}(T) = 1,008,333.33$. (There will be one month of accrued interest in 10 months time.)

Secondly, $\mathbb{A} = \mathbb{C} + \mathbb{I}_\mathbb{A}(0) = 960,000$. (There is currently three months of accrued interest.)

Thirdly, $\mathbb{F}_\mathbb{A} = \left[960,000 - 50,000\left[e^{-\frac{3}{12}9\%} + e^{-\frac{9}{12}9.5\%}\right]\right] e^{\frac{10}{12}10\%} = 939,683.97.$

Hence $V_c = 7,968.60$ and $V_p = 71,129.06$.

3.1.1 Different volatility measures

The volatility above is a price volatility measure. However, quoted volatilities are often yield volatility measures. The relationship between the various volatilities of the bond is given via Ito's lemma as

$$\sigma_\mathbb{A} = -\frac{\sigma_y y \Delta}{\mathbb{A}} \tag{3.4}$$

$$\sigma_y = -\frac{\sigma_\mathbb{A}\mathbb{A}}{y\Delta} \tag{3.5}$$

$$\sigma_\mathbb{A} = \frac{\mathbb{C}}{\mathbb{A}}\sigma_\mathbb{C} \tag{3.6}$$

How do we see this? Note that

$$dy = \mu y\, dt + \sigma_y y\, dZ$$

is the geometric Brownian motion for the yield y. Now $\mathbb{A} = f(y)$ and so

$$d\mathbb{A} = \cdots\, dt + f'(y)\sigma_y y\, dZ := \cdots\, dt + \frac{\Delta\sigma_y y}{\mathbb{A}}\mathbb{A}\, dZ$$

But, also,

$$d\mathbb{A} = \nu\mathbb{A}\, dt + \sigma_\mathbb{A}\mathbb{A}\, dZ$$

and so the result follows - except for a missing minus sign. Why?

3.2 Caplets and Floorlets

See [Hull, 2005, §26.3]. Suppose that a market participant loses money if the floating rate falls. For example, they are long a FRA. The floorlet pays off if the floating rate decreases below a predetermined minimum value, which is of course called the strike. Thus, the floorlet 'tops-up' the payment received in the FRA, if required, so that the net rate received is at least the strike. The floating rate will be the prevailing 3-month LIBOR rate, which is set at the beginning of each period, with settlement at that time by discounting, much like a FRA. Indeed, it is a FRA position that this floorlet is hedging; the FRA date schedule is being applied.

Let the floorlet rate (strike) be r_K.

Let the $T_0 \times T_1$ FRA period be of length α as usual, where day count conventions are observed. Suppose the LIBOR rate for the period, observed at the determination date is J_1. Then the payoff at the determination date is $\frac{1}{1+J_1\alpha}\alpha \max(r_K - J_1, 0)$ per unit notional; a floorlet is like a put on the interest rate. Of course the valuation is the same whether settled in advance or arrears.

In an analogous fashion to a floorlet, we have a caplet, where the payout occurs when the floating rate rises above r_K, the cap rate.

3.3 Caps and Floors

A cap is like a strip of caplets which will be used to hedge a swap. However, because swaps are settled in arrears so too is each payment in the cap strip, unlike an individual caplet. Moreover, the cap strip will have the swap date schedule applied and not the FRA date schedule. So, caps stand to swaps exactly as caplets stand to FRAs.

A cap might be forward starting or spot starting (that is, starting immediately). However, in the latter case, there is no payment in 3 months time - it is excluded from the computations and from any payments because there is no optionality. Thus, for example, a 2y cap actually has seven payments, not eight. Alternatively, one might consider that a spot starting cap is actually a forward starting cap starting in 3m time.

Let the cap rate (strike) be r_K. Let the i^{th} reset period from t_{i-1} to t_i be of length α_i as usual, where day count conventions are observed. Suppose the LIBOR rate for the period, observed at time t_{i-1}, is J_i. Then the payoff at time t_i is $\alpha_i \max(J_i - r_K, 0)$ per unit notional.

In an analogous fashion to writing a cap, we can write a floor, where the payout occurs when the floating rate drops below r_K, the floor rate.

3.3.1 Valuation

We value each caplet or floorlet separately off the yield curve using the implied forward rates at $t = 0$, for each time period t_i. Then,

$$V = \sum_{i=1}^{n} V_i \tag{3.7}$$

$$V_i = Z(0,t_i)\alpha_i\eta\left[f(0;t_{i-1},t_i)N\left(\eta d_1^i\right) - r_K N\left(\eta d_2^i\right)\right] \tag{3.8}$$

$$d_{1,2}^i = \frac{\ln\frac{f(0;t_{i-1},t_i)}{r_K} \pm \frac{1}{2}\sigma_i^2 t_{i-1}}{\sigma_i\sqrt{t_{i-1}}} \tag{3.9}$$

where $\eta = 1$ stands for a cap(let), $\eta = -1$ for a floor(let), and where the floating rate (swap) curve is being used. $f(0;t_{i-1},t_i)$ is the simple forward rate for the period from t_{i-1} to t_i. So we apply (1.11). Cap prices are quoted with $\sigma_i = \sigma$ for $i = 1, 2, \ldots, n$. On the other hand, if we are assembling a set of caplets into a cap, then the σ_i will be different.

Note that the i^{th} caplet is being valued in the t_i forward measure.

3.3.2 A call/put on rates is a put/call on a bond

A caplet (a call on an interest rate) is actually a put on a floating-rate bond whose yield is the LIBOR floating rate (a put option because of the inverse relationship between yield and bond price).

To see this, each caplet has a payoff in arrears of $\alpha_i \max(r_i - r_K, 0)$. The value of this in advance is

$$\begin{aligned}
V(t_{i-1}) &= (1+\alpha_i r_i)^{-1}\alpha_i \max(r_i - r_K, 0)\\
&= \max\left(\frac{\alpha_i r_i - \alpha_i r_K}{1+\alpha_i r_i}, 0\right)\\
&= \max\left(1 - \frac{1+\alpha_i r_K}{1+\alpha_i r_i}, 0\right)\\
&= (1+\alpha_i r_K)\max\left(\frac{1}{1+\alpha_i r_K} - \frac{1}{1+\alpha_i r_i}, 0\right)
\end{aligned}$$

which at time $t = t_0$ is $1 + \alpha_i r_K$ many puts on a zero coupon bond maturing at time t_i with the option exercise at t_{i-1}, strike $\frac{1}{1+\alpha_i r_K}$.

Likewise, floors - which are European put options on rates - are actually European call options on the (underlying) floating-rate bond.

Example 3.3.1. Suppose we have a given term structure

Term	Rate	Z
0.75	11.00%	0.92081
1	11.30%	0.89315

and consider a 9×12 caplet with strike $r_K = 12.1818\%$ and yield volatility $\sigma_y = 10\%$. Then the forward period is $\alpha = 0.25$ and the forward NACQ rate is given by $r_F = \left(\frac{0.92081}{0.89315} - 1\right)0.25 = 12.3880\%$. We find $d_1 = 0.23708$ and $d_2 = 0.15048$ using (3.9), and calculate $N(d_1) = 0.59370$ and $N(d_2) = 0.55981$. Thus, using (3.8), we obtain the value of the caplet as $V_c = 0.0011953$.

Now we consider the put on a bond. We find the strike $K = \frac{1}{1+\alpha r_K} = 0.97045$, price volatility $\sigma = \sigma_y r_F \alpha = 0.3097\%$ (using (3.4) - α is the duration of the forward bond). From (3.3), calculate

$d_1 = -0.18509$ and $d_2 = -0.18778$, and hence find $N(d_1) = 0.57342$ and $N(d_2) = 0.57447$. Then the value of the bond option from (3.2) is $V_b = 0.00116$. However, we still need to take into account the size which is given by $1 + \alpha r_K = 1.03045$. Multiplying the size with V_b yields 0.0011952.

Differences between these two values typically occur at the 5^{th} decimal place. It is impossible, mathematically, to have these two values equal; in the caplet model, rates are lognormal and in the bond option model, bond prices are lognormal. These two models cannot be made compatible.

3.3.3 Greeks

Let $\overline{r}$ denote the entire yield curve.
We need the following preliminary calculations:

$$\begin{aligned}
\frac{\partial}{\partial \overline{r}} f(0;t_{i-1},t_i) &= 1 && (3.10)\\
\frac{\partial}{\partial r} Z(0,t_i) &= -t_i Z(0,t_i) && (3.11)\\
\frac{\partial}{\partial \tau} Z(0,t_i) &= -r_i Z(0,t_i) && (3.12)\\
\frac{\partial}{\partial \overline{r}} d_1^i &= \frac{\partial}{\partial \overline{r}} d_2^i = \frac{1}{f(0;t_{i-1},t_i)\sigma_i\sqrt{t_{i-1}}} && (3.13)\\
\frac{\partial}{\partial \sigma} d_1^i &= \frac{\partial}{\partial \sigma} d_2^i + \sqrt{t_{i-1}} && (3.14)\\
\frac{\partial}{\partial t_{i-1}} d_1^i &= \frac{\partial}{\partial t_{i-1}} d_2^i + \frac{\sigma}{2\sqrt{t_{i-1}}} && (3.15)
\end{aligned}$$

pv01

$$\begin{aligned}
\frac{\partial V_i}{\partial \overline{r}} &= -t_i V_i + Z(0,t_i)\alpha_i \eta \left[\frac{\partial}{\partial \overline{r}} f(0;t_{i-1},t_i) N(\eta d_1^i)\right.\\
&\quad \left. + f(0;t_{i-1},t_i) N'(\eta d_1^i)\frac{\partial}{\partial \overline{r}} d_1^i - r_X N'(\eta d_2^i)\frac{\partial}{\partial \overline{r}} d_2^i\right]\\
&= -t_i V_i + Z(0,t_i)\alpha_i \eta N(\eta d_1^i)\left[1 + \alpha_i f(0;t_{i-1},t_i)\right] && (3.16)\\
\frac{\partial V}{\partial \overline{r}} &= \sum_{i=1}^{n} \frac{\partial V_i}{\partial \overline{r}} && (3.17)
\end{aligned}$$

Then pv01 $= \frac{1}{10000}\frac{\partial V}{\partial \overline{r}}$.

Vega

Vega is $\frac{\partial V}{\partial \sigma}$. It is the Greek w.r.t. the cap volatility; it is not a Greek with respect to the caplet volatilities, so $\sigma_i = \sigma$ for $i = 1,\ 2,\ \ldots,\ n$.

$$\begin{aligned}
\frac{\partial V_i}{\partial \sigma} &= Z(0,t_i)\alpha_i\left[f(0;t_{i-1},t_i)N'(d_1^i)\frac{\partial}{\partial \sigma}d_1^i - r_X N'(d_2^i)\frac{\partial}{\partial \sigma}d_2^i\right] \\
&= Z(0,t_i)\alpha_i f(0;t_{i-1},t_i)N'(d_1^i)\sqrt{t_{i-1}} \qquad (3.18) \\
\frac{\partial V}{\partial \sigma} &= \sum_{i=1}^{n}\frac{\partial V_i}{\partial \sigma} \qquad (3.19)
\end{aligned}$$

Bucket (Caplet/Floorlet) Vega

For sensitivities to caplet volatilities, we would be looking at a bucket risk type of scenario. This would be $\frac{\partial V}{\partial \sigma_i}$, and this only makes sense of course in the case where we have individual forward-forward volatilities rather than just a flat cap volatility.

$$\frac{\partial V_i}{\partial \sigma_i} = Z(0,t_i)\alpha_i f(0;t_{i-1},t_i)N'(d_1^i)\sqrt{t_{i-1}}$$

where now it is the caplet volatility σ_i being used, not the cap volatility σ.

Theta

Theta can only be calculated w.r.t. some explicit assumptions about yield curve evolution. The assumption can be that when time moves forward, the bootstrapped continuous curve will remain constant. By full revaluation we can then calculate the theta of the position with one day of time decay.

Delta hedging caps/floors with swaps

We require the 'with delta' value of a cap or floor. Often the client wants to do the deal 'with delta', which means that the 'linear' hedge comes with the option trade. Thus we quote delta based on hedging the cap/floor with a (forward starting) swap with the same dates, basis and frequency. Then

$$\Delta = \frac{\frac{\partial V(\text{cap})}{\partial \underline{r}}}{\frac{\partial V(\text{swap})}{\partial \underline{r}}} = \frac{\text{pv01(cap)}}{\text{pv01(swap)}} \qquad (3.20)$$

The numerator is found in (3.17). For the denominator: the value of the swap fixed receiver, fixed rate R having been set, is

$$V(\text{swap}) = R\sum_{i=1}^{n}\alpha_i Z(0,t_i) - Z(0,t_0) + Z(0,t_n) \qquad (3.21)$$

as seen in (3.30), (3.31). So the derivative is

$$\frac{\partial V(\text{swap})}{\partial \underline{r}} = -R\sum_{i=1}^{n}\alpha_i t_i Z(0,t_i) + t_0 Z(0,t_0) - t_n Z(0,t_n) \qquad (3.22)$$

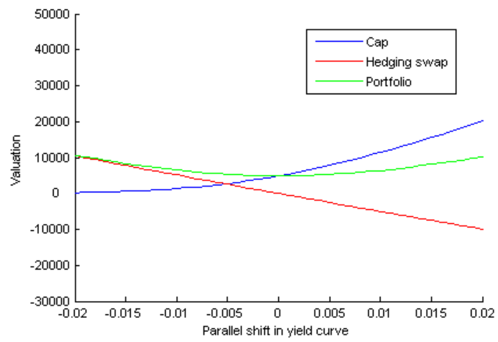


Figure 3.1: A cap with its forward swap hedge

3.4 Stripping Black caps into caplets

Since each caplet is valued separately we expect a different volatility measure for each. Cap volatilities are always quoted as flat volatilities where the same volatility is used for each caplet, which in some sense will be a weighted average of the individual caplet volatilities. Thus, we use $\sigma = \sigma_i$ for $i = 1, 2, \ldots, n$. Most traders work with independent volatilities for each caplet, though, and these are called forward-forward vols. There exists a hump at about 1 year for the forward-forward vols (and, consequently, also for the flat vol, which can be seen as a cumulative average of the forward-forward vols). This can be observed or backed out of cap prices: see Figure 3.2.

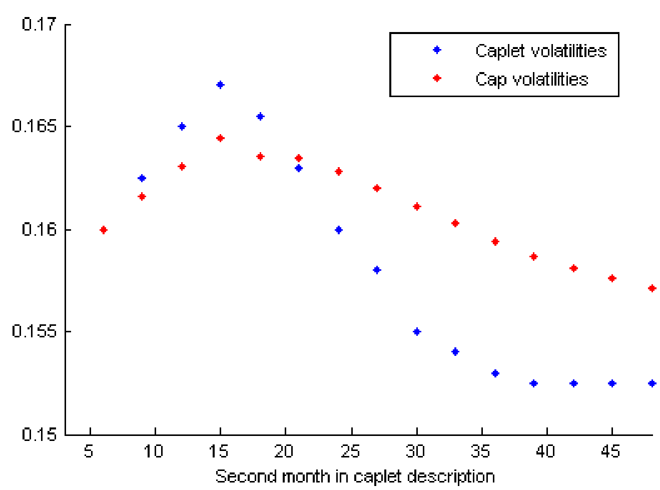


Figure 3.2: Caplet (fwd-fwd) and cap (flat fwd-fwd) volatility term structure

Thus, from a set of caplet volatilities $\sigma_1, \sigma_2, \ldots, \sigma_n$ we may need to determine the corresponding cap volatility σ. This of course is a uniquely determined implied volatility problem. It only makes sense if the strikes of all the caplets are equal to the strike of the cap; this won't be the case in general (typically quoted volatilities will be at the money).

A far more difficult problem is the specification of the term structure of caplet volatilities given an incomplete term structure of cap volatilities. This is a type of bootstrap problem: there is insufficient information to determine a unique solution. For example, only 3x6, 6x9 and 9x12 caplets and 1y, 2y, 3y , 4y caps might be available. This critical point is generally not really well dealt with in the textbooks.

Instantaneous forward rate volatilities will be specified. Suppose the instantaneous volatility of $F_k(t)$ is modelled as $\sigma_k(t)$. Having done so, one now has the implied volatility of the $T_{k-1} \times T_k$ caplet given by

$$\sigma_{k,\text{imp}} = \sqrt{\frac{1}{T_{k-1}} \int_0^{T_{k-1}} \sigma_k(t)^2 \, dt} \tag{3.23}$$

Some forms will allow us to calibrate to given cap or caplet term structures exactly. Clearly, and as emphasised in [Brigo and Mercurio, 2006, §6.2], the pricing of caps is independent of the joint dynamics of forward rates. However, that does not mean that calibration should also be an independent process. There are such straightforward formulations of the calibration of caps, so then the only parameters left to tackle swaptions calibration are the instantaneous correlations of forward rates, and this will typically be inadequate for use in the LMM.

[Brigo and Mercurio, 2006, §6.3.1] discuss seven different formulations for calibrating cap term structures which allow for more or less flexibility later in the swaption calibration.

The approach suggested in [Rebonato, 2002, Chapter 6], [Rebonato, 2004, Chapter 21] is to specify a parametric form for the instantaneous volatility such as

$$\sigma_k(t) = (a + b(T_{k-1} - t))e^{-c(T_{k-1}-t)} + d \tag{3.24}$$

This form is flexible enough to reproduce the typical shapes that occur in the market. It can accommodate either a humped form or a monotonically decreasing volatility. The model is time homogeneous, and the parameters have some economic interpretation, as described in Rebonato [2002]. For example, the time-0 volatility is $a + d$ and the long run limit is d. Furthermore, within the context of models such as LMM, calculus is easy enough: for example

$$\begin{aligned}
&\int \left((a + b(T_i - t))e^{-c(T_i-t)} + d\right) \left((a + b(T_j - t))e^{-c(T_j-t)} + d\right) \, dt \\
&= \frac{ad}{c}\left(e^{c(t-T_i)} + e^{c(t-T_j)}\right) + d^2 t - \frac{bd}{c^2}\left(e^{c(t-T_i)}\left[c(t - T_i) - 1\right] + e^{c(t-T_j)}\left[c(t - T_j) - 1\right]\right) \\
&\quad + \frac{e^{c(2t-T_i-T_j)}}{4c^3}\left[2a^2c^2 + 2abc(1 + c(T_i + T_j - 2t)) + b^2(1 + 2c^2(t - T_i)(t - T_j) + c(T_i + T_j - 2t))\right] \\
&=: I(t, T_i, T_j)
\end{aligned} \tag{3.25}$$

as in [Rebonato, 2002, §6.6 - correcting for the typo], [Jäckel, 2002, (12.13)]. Then as in (3.23)

$$\sigma_{k,\text{imp}} = \sqrt{\frac{1}{T_{k-1}} \left(I(T_{k-1}, T_{k-1}, T_{k-1}) - I(0, T_{k-1}, T_{k-1})\right)} \tag{3.26}$$

Now the problem has gone from being under-specified to over-specified; an error minimisation algorithm will be used. Financial constraints are that $a + d > 0$, $c > 0$, $d > 0$. What we do here is, for any choices of a, b, c and d,

- determine the caplet volatilities for every caplet using (3.26).
- find the model price of all the caplets using the caplet pricing formula.
- find the model price of the caps that are trading in the market (for example, the 1y, 2y, ... caps).
- We can then formulate an error function which measures the difference between model and market prices. This function will be something like

$$\mathrm{err}_{a,b,c,d} = \sum_i |V_i(a,b,c,d) - P_i| \tag{3.27}$$

  The i varies only over those caps that actually trade (are quoted) in the market.
- Minimise the error. Solver might be used, but it might need to be trained to find a reasonable solution. Use of Nelder-Mead is suggested.

Some care needs to be taken here. The inputs will be at the money cap volatilities. For each cap, the at the money level (the forward swap rate) will probably be different. These are the forwards that need to be used in the cap pricing formula. The output is a parametric form for at the money caplet volatilities, and for each of these the at the money level will be different. This difference is usually ignored, as we are pricing without any skew anyway.

Having found the parameters, one still wants to price instruments that trade in the market exactly. Thus, after the parameters $a,\ b,\ c,\ d$ have been found, the model is re-specified as

$$\sigma_k(t) = K_k \left[(a + b(T_{k-1} - t))e^{-c(T_{k-1}-t)} + d\right] \tag{3.28}$$

Equivalently,

$$\sigma_{k,\mathrm{imp}} = K_k \sqrt{\frac{1}{T_{k-1}} \left(I(T_{k-1}, T_{k-1}, T_{k-1}) - I(0, T_{k-1}, T_{k-1})\right)} \tag{3.29}$$

We assume that K_k is a piecewise constant function, changing only at the end of each cap i.e. as a cap terminates and a new calibrating cap is applied. For example, if we have a 1y and a 2y cap (and others of later tenor) then $K_2 = K_3 = K_4$, and $K_5 = K_6 = K_7 = K_8$.

With these assumptions, K_k is found uniquely. For caplets, the value of K_k is found directly. For caps, we note that there is one root find for each set of equal K_k's; we proceed from smallest to largest k. Thus

- For any given K_k, calculate the volatility using (3.29).
- price all the caplets using the caplet pricing formula.
- find the model price of the cap.
- vary K_k to match this model price with the market price. As the model price is an increasing function of K_k, the root is unique. We use a root finder such as Brent's method.

The model is no longer time homogeneous, and the deviation from being so is in some sense measured by how far the K_k deviate from 1. The better the fit of the model, the closer these values are to 1, one would hope to always have values of K_k between 0.9 and 1.1 say. This correction is discussed in [Rebonato, 2004, §21.4].

3.5 Swaptions

A swaption is an option to enter into a swap. We consider European swaptions. (Bermudan swaptions also exist.) Thus, at a specified time t_0, the holder of the option has the option to enter a swap which commences then (the first payment being one time period later, at t_1, and lasts until time t_n).

Of course, we have two possibilities

(a) a payer swaption, which gives the holder the right but not the obligation to receive floating, and pay a fixed rate r_K (a call on the floating rate).

(b) a receiver swaption, which gives the holder the right but not the obligation to receive a fixed rate r_K, and pay floating (a put on the floating rate).

Let f be the fair (par) forward swap rate for the period from t_0 to t_n. The date schedule for swaptions is the swap schedule. The time of payments of the forward starting swap are $t_1,\ t_2,\ \ldots,\ t_n$, where $t_0,\ t_1,\ \ldots,\ t_n$ are successive observation days, for example, quarterly, calculated according to the relevant day count convention and modified following rules. As usual, let $t_i - t_{i-1} = \alpha_i$, measured in years, for $i = 1,\ 2,\ \ldots,\ n$.

Note that if $n = 1$ then we have a one period cap (a payer swaption) or a one period floor (a receiver swaption). Thus, modulo the date schedule and the advanced/arrears issue, a caplet or floorlet.

Note that in general a swap (forward starting or starting immediately; in the later case $t_0 = 0$) with a fixed rate of R has the fixed leg payments worth

$$V_{\text{fix}} = R \sum_{i=1}^{n} \alpha_i Z(0, t_i) \tag{3.30}$$

while the floating payments are worth

$$V_{\text{float}} = Z(0, t_0) - Z(0, t_n) \tag{3.31}$$

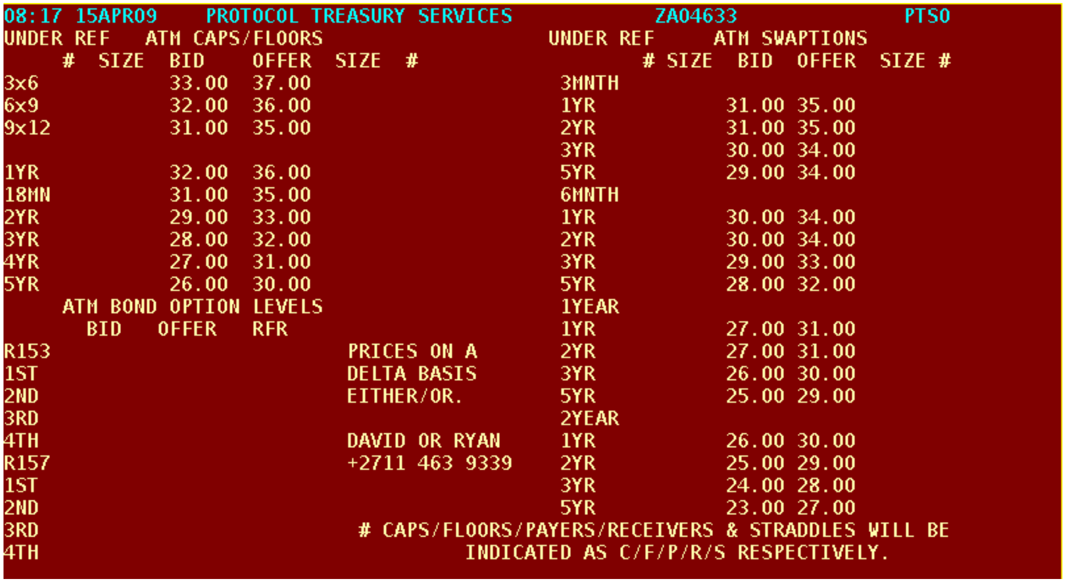
Hence the fair forward swap rate, which equates the fixed and floating leg values, is given by

$$f = \frac{Z(0, t_0) - Z(0, t_n)}{\sum_{i=1}^{n} \alpha_i Z(0, t_i)} \tag{3.32}$$

Of course, these values are derived from the existing swap curve. Thus, the fair forward swap rate is dependent upon the bootstrap and interpolation method associated with the construction of the yield curve. Nevertheless, empirically it is found that the choice of interpolation method will only affect the result to less than a basis point, and typically a lot less. Also, let

$$L = \sum_{i=1}^{n} \alpha_i Z(0, t_i) \tag{3.33}$$

L is called the level, or the annuity.



```
08:17 15APR09     PROTOCOL TREASURY SERVICES            ZA04633              PTSO
UNDER REF    ATM CAPS/FLOORS                        UNDER REF      ATM SWAPTIONS
     #  SIZE  BID     OFFER  SIZE  #                        # SIZE  BID  OFFER  SIZE #
3x6           33.00  37.00                           3MNTH
6x9           32.00  36.00                           1YR           31.00 35.00
9x12          31.00  35.00                           2YR           31.00 35.00
                                                     3YR           30.00 34.00
1YR           32.00  36.00                           5YR           29.00 34.00
18MN          31.00  35.00                           6MNTH
2YR           29.00  33.00                           1YR           30.00 34.00
3YR           28.00  32.00                           2YR           30.00 34.00
4YR           27.00  31.00                           3YR           29.00 33.00
5YR           26.00  30.00                           5YR           28.00 32.00
     ATM BOND OPTION LEVELS                          1YEAR
       BID    OFFER    RFR                           1YR           27.00 31.00
R153                             PRICES ON A         2YR           27.00 31.00
1ST                              DELTA BASIS         3YR           26.00 30.00
2ND                              EITHER/OR.          5YR           25.00 29.00
3RD                                                  2YEAR
4TH                              DAVID OR RYAN       1YR           26.00 30.00
R157                             +2711 463 9339      2YR           25.00 29.00
1ST                                                  3YR           24.00 28.00
2ND                                                  5YR           23.00 27.00
3RD                               # CAPS/FLOORS/PAYERS/RECEIVERS & STRADDLES WILL BE
4TH                                        INDICATED AS C/F/P/R/S RESPECTIVELY.
```

Figure 3.3: A Reuter's page for at the money volatility quotes for caps/floors, caplet/floorlets, and swaptions. In the swaption table 3mnth, 6mnth, 1year, 2year refers to the expiry date of the option; 1yr, 2yr, 3yr, 5yr refers to the tenor of the swap.

3.5.1 Valuation

The value of the swaption per unit of nominal is [Hull, 2005, §26.4]

$$\begin{aligned} V_\eta &= L\eta[fN(\eta d_1) - r_K N(\eta d_2)] && (3.34) \\ d_{1,2} &= \frac{\ln \frac{f}{r_K} \pm \frac{1}{2}\sigma^2 t_0}{\sigma\sqrt{t_0}} && (3.35) \end{aligned}$$

where $\eta = 1$ stands for a payer swaption, $\eta = -1$ for a receiver swaption. Here σ is the volatility of the fair forward swap rate, and is an implied variable quoted in the market.

3.5.2 Greeks

Delta

$$\Delta = \eta N(\eta d_1) \tag{3.36}$$

pv01

This is given by

$$\begin{aligned} \frac{\partial V}{\partial \bar{r}} &= \eta \frac{\partial L}{\partial \bar{r}}\left[fN(\eta d_1) - r_K N(\eta d_2)\right] + \eta L\left[\frac{\partial f}{\partial \bar{r}}N(\eta d_1) + fN'(\eta d_1)\eta\frac{\partial d_1}{\partial r} - r_K N'(\eta d_2)\eta\frac{\partial d_2}{\partial r}\right] \\ &= \eta \frac{\partial L}{\partial \bar{r}}\left[fN(\eta d_1) - r_K N(\eta d_2)\right] + \eta L\frac{\partial f}{\partial \bar{r}}N(\eta d_1) && (3.37) \end{aligned}$$

We now apply several times the fact that

$$\frac{\partial}{\partial \overline{r}} Z(0,t) = -tZ(0,t) \tag{3.38}$$

Firstly

$$\frac{\partial L}{\partial \overline{r}} = \sum_{i=1}^{n} -t_i \alpha_i Z(0,t_i) \tag{3.39}$$

$$\tag{3.40}$$

Also, if we have a function $\frac{g(\overline{r},\tau)}{h(\overline{r},\tau)}$, then

$$\left(\frac{g}{h}\right)' = g'h^{-1} - gh^{-2}h' \tag{3.41}$$

$$\tag{3.42}$$

where the differentiation is with respect to $\overline{r}$. We apply (3.38) and this to (3.32).
Then pv01 $= \frac{1}{10000}\frac{\partial V}{\partial \overline{r}}$.

Vega

$$\begin{aligned} \frac{\partial V}{\partial \sigma} &= L\eta\left[fN'(\eta d_1)\eta \frac{\partial d_1}{\partial \sigma} - r_K N'(\eta d_2)\eta \frac{\partial d_2}{\partial \sigma}\right] \\ &= LfN'(d_1)\sqrt{\tau} \end{aligned} \tag{3.43}$$

Theta

As before.

3.5.3 Why Black is useless for exotics

In each Black model for maturity date T, the forward rates will be log-normally distributed in what is called the T-forward measure, where T is the pay date of the option. The fact that we can 'legally' discount the expected payoff under this measure using today's yield curve is a consequence of some profound academic work of Geman et al. [1995], which establishes the existence of alternative pricing measures, and the ways that they are related to each other. This paper is very significant in the development of the Libor Market Model.

We can use Black's model without knowing any of the theory of Geman et al. [1995]; however, the Black model cannot safely be used to value more complicated products where the payoff depends on observations at multiple dates. For this an alternative model which links the behaviour of the rates at multiple dates will need to be used.

For this, the most extensive approach is the Libor Market Model. Here inputs are all the Black models as well as a correlation structure between all the forward rates. From a properly calibrated LMM, one recaptures (up to the calibration error) the prices of traded caplets, caps, and swaptions. However, this calibration can be difficult. It then requires Monte Carlo techniques to value other derivatives.

An intermediate approach is the use of a more parsimonious model with just a few driving factors - the so-called single-factor models.

3.6 Exercises

1. A company caps three-month JIBAR at 9% per annum. The principal amount is R10 million. On a reset date, namely 20 September 2004, three-month JIBAR is 10% per annum.

   (a) What payment would this lead to under the cap?

   (b) When would the payment be made?

   (c) What is the value of the payment on the reset date?

2. Use Black's model to value a one-year European put option on a bond with 9 years and 11 months to expiry. Assume that the current cash price of the bond is R105, the strike price (clean) is R110, the one-year interest rate is 10%, the bond's price volatility measure is 8% per annum, and the coupon rate is 8% NACS.

3. Consider an eight-month European put option on a Treasury bond that currently has 14.25 years to maturity. The current cash bond price is \$910, the exercise price is \$900, and the volatility measure for the bond price is 10% per annum. A semi-annual coupon of \$35 will be paid by the bond in three months. The risk-free interest rate is 8% for all maturities up to one year.

   (a) Use Black's model to determine the price of the option. Consider both the case where the strike price corresponds to the cash price of the bond and the case where it corresponds to the clean price.

   (b) Calculate delta, gamma (both with respect to the bond price B) and vega in the above problem when the strike price corresponds to the quoted price. Explain how they can be interpreted.

4. Using Black's model calculate the price of a caplet on the JIBAR rate. Today is 20 September 2004 and the caplet is the one that corresponds to the 9x12 period.

   The caplet is struck at 9.4%. The current JIBAR rate is 9.3%, the 3x6 FRA is 9.4% and the 6x9 FRA is 9.34%, and the 9x12 FRA is 9.20%.

   Interest-rate volatility is 15%.

   Also calculate delta, gamma (both with respect to the 9x12 forward rate) and vega in the above problem.

5. Suppose that the yield, R, on a discount bond follows the process

   $$dR = \mu(R, t)dt + \sigma(R, t)dz$$

   where dz is a standard Wiener process under some measure. Use Ito's Lemma to show that the volatility of the discount bond price declines to zero as it approaches maturity, irrespective of the level of interest rates.

6. The price of a bond at time T, measured in terms of its yield, is $G(y_T)$. Assume geometric Brownian motion for the forward bond yield, y, in a world that is forward risk-neutral with respect to a bond maturing at time T. Suppose that the growth rate of the forward bond yield is α and its volatility is σ_y.

(a) Use Ito's Lemma to calculate the process for the forward bond price, in terms of α, σ_y, y and $G(y)$.

(b) The forward bond price should follow a martingale in the world we are considering. Use this fact to calculate an expression for α.

(c) Assume an initial value of $y = y_0$. Now show that the expected value of y at time T can be directly calculated from the above expression.

7. Consider the following quarterly data:

	1	2	3	4	5	6	7	8
zero	8.00%	8.50%	8.30%	8.00%	8.00%	8.20%	8.25%	8.40%
vol	17.00%	16.00%	17.00%	16.00%	15.00%	15.00%	14.00%	

Time here is modified following quarterly.

The volatility data represents the volatility of the all in price for a bond option on the r153 that expires at that time.

The details of the r153 are as follows:

Bond Name	Maturity	Coupon	BCD1	BCD2	CD1	CD2
R153	2010/08/31	13.00%	821	218	831	228

(The BCD details of all r bonds have changed.)

Today is 29 August 2004 and the all in price is 1.1010101.

Price a vanilla European bond option, with a clean price strike, with expiry 30 May 2005, according to Black's model.

Construct your yield curve using raw interpolation.

Chapter 4

One and two factor continuous-time interest rate models

This chapter is derived from components of [Wilmott, 2000, Chapters 40, 41], Svoboda [2002], Svoboda [2003]; amongst others.

The fundamental complicating factor in interest rate models is the non-traded nature of rates. Coupled to this is the non-linear, inverse relationship between bond prices and yields. If a derivative is dependent on one or more interest rates, the rather neat consequences of the Black-Scholes model for equity derivatives, where the expected rate of return μ drops out of the pricing formula and is 'replaced' by risk-neutral valuation and a return of r, will almost certainly not be valid. What is more, in that Black-Scholes differential equation it was exactly that r which was a constant, it is now a variable.

We develop models of the short rate. The short rate will be denoted r. The short rate itself is quite a theoretical concept: at time t, $r(t)$ is the yield on a bond which matures at time $t + dt$. In practice, a rate that truly exists, such as the overnight, one month or even three month rate, will be used as a surrogate for this rate.

We will now have a new variable, known as the market price of risk; knowing the market price of risk is equivalent to knowing the expected rate of return.

4.1 Derivatives Modelled on a Single Stochastic Variable

Assume that the short rate can be modelled as an Itô process of the following kind:

$$dr = \mu(r,t)dt + \sigma(r,t)dz \tag{4.1}$$

where dz has the usual properties. (4.1) is sufficiently general for a broad spectrum of possible models.

Recall that if $V = V(r,t)$ is a sufficiently well behaved function then Itô's lemma tells us

$$\begin{aligned} dV &= \frac{\partial V}{\partial r}\, dr + \left(\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2}\right) dt \\ &= \left(\frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\mu + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2}\right) dt + \frac{\partial V}{\partial r}\sigma\, dz \\ &:= \mu_V V\, dt + \sigma_V V\, dz \end{aligned}$$

Assume that V_1 and V_2 are sufficiently well behaved. The underlying stochastic process r is non-traded, so the only possible way of creating a riskless portfolio is by using both V_1 and V_2. Let Π be the portfolio which is long 1 of instrument 1 and is short $\frac{\sigma_1 V_1}{\sigma_2 V_2}$ of instrument 2. Then, the risky component (the part with dz) has been eliminated, so as usual

$$\begin{aligned} \mu_1 V_1 - \mu_2 \frac{\sigma_1 V_1}{\sigma_2 V_2} V_2 &= r\left(V_1 - \frac{\sigma_1 V_1}{\sigma_2 V_2} V_2\right) \\ \mu_1 - \mu_2 \frac{\sigma_1}{\sigma_2} &= r\left(1 - \frac{\sigma_1}{\sigma_2}\right) \\ \frac{\mu_1 - r}{\sigma_1} &= \frac{\mu_2 - r}{\sigma_2} \end{aligned} \tag{4.2}$$

and so the quantity

$$\lambda := \frac{\mu_V - r}{\sigma_V} \tag{4.3}$$

is independent of the choice of bond, and called the market price of risk. Now

$$\mu_V V - \lambda \sigma_V V = rV$$

and so

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}\mu + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - \lambda \frac{\partial V}{\partial r}\sigma &= rV \\ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial r}(\mu - \lambda\sigma) + \tfrac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial r^2} - rV &= 0 \end{aligned} \tag{4.4}$$

Note well - the drift, volatility here are those of (4.1), and not of (4.2).

This is the Black-Scholes type equation for interest rate derivatives when the yield curve is modelled by a one-factor process of the type (4.1). Note that a vanilla bond is a derivative; options on bonds are derivatives of derivatives. The underlying here is the short rate.

It is possible to apply (4.4) to bonds that have coupons by adding $K(r,t)$ to the left-hand side, where $K(r,t)\, dt$ represents the amount of coupon received in the period dt. (This may be continuous or discrete; in the latter case we will be using a finite difference scheme with specified jumps in value to the bond as it goes ex coupon.)

We require two "spatial" boundary conditions and one final boundary condition to fully specify the model. For example, we know that if $V(r,t,T)$ is the price of a bond with par value 1, $V(r,T,T) = 1$. Other examples would be the terminal intrinsic value of a vanilla option, etc.

Using the Feynman-Kac theorem (4.4) is the solution to the expectation of the r-discounted value of $V(T)$ where r is subject to the Itô process

$$dr = (\mu - \lambda\sigma)dt + \sigma d\overline{z} \tag{4.5}$$

The Brownian motion here is not the same as the original one: $d\bar{z} = \lambda dt + dz$. The drift here is $\mu - \lambda\sigma$; the original (real world) drift was of course μ.

4.2 What is the market price of risk?

The market price of risk is the return in excess of the risk-free rate that the market wants as compensation for taking risk.

In classical economic theory no rational person would invest in a risky asset unless they expect to beat the return from holding a risk-free asset. Classically risk is measured by volatility. The market price of risk thus has the form of (4.3). This quantity is not affected by leverage.

If the modelled quantity is directly traded, then one can continuously and perfectly hedge a position, and in so doing eliminate risk totally. Thus the market price of risk is a quantity that does not need to appear in the model (or the differential equation that describes it).

If the modelled quantity is not directly traded, then there will be an explicit reference in the option-pricing model to the market price of risk. This is because associated risk cannot be hedged, and so how much extra return is needed to compensate for taking this unhedgeable risk must be modelled. When a quantity that is not traded is modelled stochastically then the equation governing the pricing of derivatives is usually of diffusion form, with the market price of risk appearing in the drift term with respect to the non-traded quantity.

An example is the interest rate market. Since interest rates are not tradable, you will not be able to create a risk-free portfolio. In fact you will trade one or more assets that depends on that quantity, rather than the quantity itself (e.g. you want to take a position on an interest rate, you might want to trade bonds since their value depends on interest rates). The same happens for hedging, since you will need again to use another instrument similar to the original one to cover your position, but is not the underlying quantity.

The result is that there is never have a portfolio that completely eliminates risk, and so an agent will require a premium to balance his diminished utility function resulting from taking risk.

Another example is a stochastic volatility model in the equity market, such as Heston. Here the volatility is a risk that is not a traded asset. However, the market can be completed by trading in any vanilla option. This was observed in Hagan et al. [2002], and formalised in Davis and Obłój [2007]. It can also be completed by trading in a variance swap.

4.3 Exogenous (equilibrium) short rate models

These early equilibrium models are based on a mathematical model of the economy. They focus on describing and explaining the interest rate term structure. However, a fundamental problem with the equilibrium approach is that the models may not be arbitrage free, in other words, they fail to price even the vanilla inputs trading in the market. This is to be expected: these instruments are not inputs to the models, they are outputs. As a consequence it is unlikely that these models will be used nowadays.

4.3.1 GBM model

Dothan [1978], Rendelman and Bartter [1980]
This is simply

$$dr = \mu r dt + \sigma r dz \tag{4.6}$$

which is the same as the usual log-normal stock price model. However, an obvious problem that arises is that of mean-reversion: interest rates appear to be pulled back to some long-run average level over time.

4.4 A particular class of models

(4.4) is the bond pricing equation for an arbitrary model. In addition there will be the boundary conditions as already mentioned. In order to progress, we want to specify the risk neutral drift $\mu - \lambda\sigma$ and the volatility σ to arrive at tractable models i.e. models where (at the very least) the price of zero coupon bonds can be found analytically.[1] In other words, we will attempt to solve prices under the process (4.5).
Time will be denoted by the variable t; it starts at time 0 and runs to terminal time T. Calibration takes place at time 0; T is the maturity of the longest dated bond under consideration (or even further, for that matter). $Z(r, t, T)$ is the value at time t of a zero coupon bond maturing at time T. The only stochastic variable is $r = r(t)$ i.e. the yield curve at time t is determined entirely by the value of the short rate $r(t)$.
The class of models we consider is as follows: the zero-coupon bond value has dependency on the one-factor short rate via:

$$Z(r, t, T) = \exp(A(t, T) - rB(t, T)) \tag{4.7}$$

The A and B functions are functions of t; they are determined at time 0. The class of models that result from this assumption are referred to as affine. The different affine models will have different formulae for $A(t, T)$ and $B(t, T)$. The continuous yield curve at time t is determined as follows:

$$\mathbb{r}(r, t, T) = -\frac{A(t, T) - rB(t, T)}{T - t} \tag{4.8}$$

So, if the short rate r varies instantaneously, the entire yield curve $\mathbb{r}(r, t, T)$ varies.
Note that

$$\begin{aligned}
\frac{\partial Z}{\partial t} &= Z\left(\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t}\right)\\
\frac{\partial Z}{\partial r} &= -B(t, T)Z\\
\frac{\partial^2 Z}{\partial r^2} &= B(t, T)^2 Z
\end{aligned}$$

The bond described in (4.7) is a derivative of the short rate, so it satisfies (4.4). Dividing by Z we get

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \tfrac{1}{2}\sigma^2 B^2 - (\mu - \lambda\sigma)B - r = 0. \tag{4.9}$$

[1] This will give us a check on the model, by comparing these results to our bootstrap, and in fact, might turn the whole problem inside-out, and become the calibration procedure.

If we differentiate this equation twice with respect to r, and divide through by B, we arrive at

$$\tfrac{1}{2}B\frac{\partial^2}{\partial r^2}\sigma^2 - \frac{\partial^2}{\partial r^2}(\mu - \lambda\sigma) = 0.$$

In this the expiry date T is arbitrary, so the coefficients must be separately zero:

$$\frac{\partial^2}{\partial r^2}\sigma^2 = 0$$
$$\frac{\partial^2}{\partial r^2}(\mu - \lambda\sigma) = 0$$

and so we can find functions $\alpha(t)$, $\beta(t)$, $\theta(t)$ and $\gamma(t)$ which are function of t alone (and not of r) such that

$$\sigma^2(r,t) = \alpha(t)r + \beta(t) \tag{4.10}$$
$$\mu(r,t) - \lambda(r,t)\sigma(r,t) = \theta(t) - \gamma(t)r \tag{4.11}$$

Substituting (4.10) and (4.11) into (4.9) we have

$$\frac{\partial A}{\partial t} - r\frac{\partial B}{\partial t} + \tfrac{1}{2}(\alpha(t)r + \beta(t))B^2 - (\theta(t) - \gamma(t)r)B - r = 0$$

and so

$$\frac{\partial A}{\partial t} = \theta(t)B - \tfrac{1}{2}\beta(t)B^2 \tag{4.12}$$
$$\frac{\partial B}{\partial t} = -1 + \gamma(t)B + \tfrac{1}{2}\alpha(t)B^2 \tag{4.13}$$

(4.13) is what is known as a Ricatti equation, it is an equation in B which does not involve A. Having solved (4.13) we then insert this solution into (4.12) and do straightforward integration to obtain A. In general the system will be solved numerically; however, sometimes it is possible to find closed form solutions, and we will focus on these cases.

The boundary conditions must be:

$$A(T,T) = 0$$
$$B(T,T) = 0$$

The process in (4.5) has become

$$dr = (\theta(t) - \gamma(t)r)dt + \sqrt{\alpha(t)r + \beta(t)}dz \tag{4.14}$$

4.4.1 Constant parameter model

Let us start with the simplest case: all of α, β, θ, γ are constants. Then

$$\frac{dB}{dt} = -1 + \gamma B + \tfrac{1}{2}\alpha B^2$$

and so (after some unpleasant calculus)

$$B(t,T) = \frac{2(\exp(\psi_1(T-t)) - 1)}{(\gamma + \psi_1)(\exp(\psi_1(T-t)) - 1) + 2\psi_1}$$
$$\psi_1 = \sqrt{\gamma^2 + 2\alpha}$$

Now, we can write

$$\frac{dA}{dB} = \frac{\theta B - \frac{1}{2}\beta B^2}{\frac{1}{2}\alpha B^2 + \gamma B - 1}$$

and so

$$\begin{aligned}
\tfrac{1}{2}\alpha A &= a\psi_2 \ln(a - B) + (\psi_2 + \tfrac{1}{2}\beta) b \ln \frac{B + b}{b} - \tfrac{1}{2}\beta B - a\psi_2 \ln a \\
\psi_2 &= \frac{\theta - \frac{1}{2}a\beta}{a + b} \\
b, a &= \frac{\pm\gamma + \sqrt{\gamma^2 + 2\alpha}}{\alpha}
\end{aligned}$$

Note that here we have incorporated the final conditions.
Obviously A and B are functions of $\tau = T - t$, and not functions of t and T separately.

4.4.2 Vasicek model

Vasicek [1977] models the short rate as an Ornstein-Uhlenbeck process; the mean-reversion property is modelled. All four parameters are constant, with $\alpha = 0$, and (of necessity) $\beta > 0$. Thus the volatility is time and state independent and equal to $\sqrt{\beta}$. So

$$\begin{aligned}
dr &= (\theta - \gamma r)dt + \sqrt{\beta} dz \\
&= \gamma \left(\frac{\theta}{\gamma} - r \right) dt + \sqrt{\beta} dz \\
&:= \gamma \left(\mu - r \right) dt + \sigma dz
\end{aligned} \tag{4.15}$$

The short rate mean reverts to μ at a 'speed' of γ. This model is very tractable, and there are explicit solutions for a number of derivatives based on it. Unfortunately, the Vasicek model permits negative interest rates.
Now (4.12) and (4.13) become

$$\begin{aligned}
\frac{dA}{dt} &= \theta B - \tfrac{1}{2}\beta B^2 \\
\frac{dB}{dt} &= -1 + \gamma B
\end{aligned}$$

(there is no dependency on r in the parameters - t is the only variable in play, and partial differentiation is the same as total differentiation) and so (again, after some work)

$$\begin{aligned}
B(t, T) &= \frac{1}{\gamma}(1 - \exp(-\gamma(T - t))) \\
A(t, T) &= \frac{(\frac{1}{2}\beta - \theta\gamma)((T - t) - B)}{\gamma^2} - \frac{\beta B^2}{4\gamma}
\end{aligned}$$

It is possible to generate increasing, decreasing or humped curves under this model. See Figure 4.1.
We can solve for the distribution of the short rate. First, put $f(r, t) = re^{\gamma t}$ - here we are using an

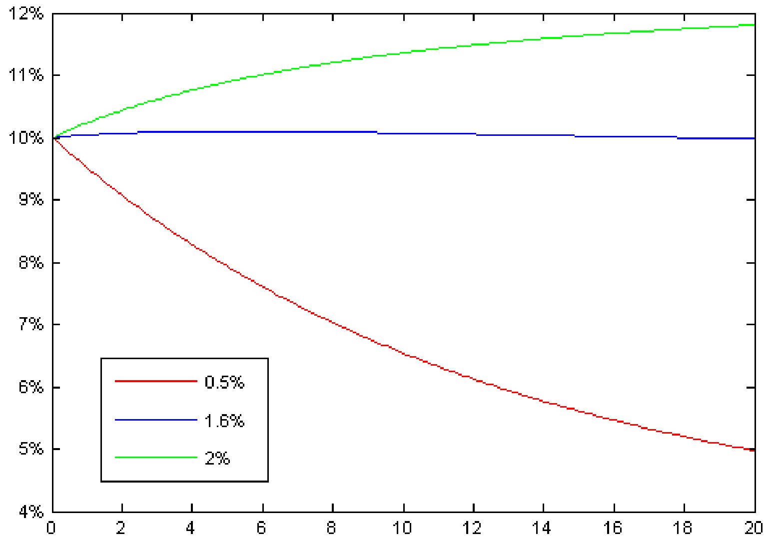


Figure 4.1: Possible yield curves under the Vasicek model, varying only θ.

integrating factor. Using Itô's lemma, we have

$$\begin{aligned}
df &= e^{\gamma t}\, dr + \gamma e^{\gamma t} r\, dt \\
&= e^{\gamma t}\, \left[\gamma\left(\mu - r\right) dt + \sigma dz\right] + \gamma e^{\gamma t} r\, dt \\
&= e^{\gamma t}\gamma\mu dt + \sigma e^{\gamma t}\, dt \\
f(t) - f(0) &= \gamma\mu \int_0^t e^{\gamma s}\, ds + \sigma \int_0^t e^{\gamma s} dz(s) \\
r_t e^{\gamma t} - r_0 &= \mu\left[e^{\gamma t} - 1\right] + \sigma \int_0^t e^{\gamma s} dz(s) \\
r_t &= r_0 e^{-\gamma t} + \mu\left[1 - e^{\gamma t}\right] + \sigma \int_0^t e^{\gamma(s-t)} dz(s)
\end{aligned}$$

and so

$$r_t \sim \phi\left(\mu + e^{-\gamma t}(r_0 - \mu), \frac{\sigma^2}{2\gamma}\left(1 - e^{-2\gamma t}\right)\right) \tag{4.16}$$

In particular, it is possible to obtain negative rates under this model.

Under the Vasicek model we can value European options on default-free bonds exactly Jamshidian [1989]. Consider a call option on an T_2-maturity discount bond with exercise price K and expiration $T_1 < T_2$. Using risk-neutral valuation, the vanilla option price is:

$$V_{\pm}[r, t, T_1, T_2; K] = Z(r, t, T_1)\mathbb{E}_t^{\mathbb{Q}}[\max(0, \pm(Z - K))]$$

where Z is the zero coupon price for expiry T_2 as observed at time T_1. It has current forward value

$$f(Z) = \frac{Z(r, t, T_2)}{Z(r, t, T_1)}.$$

It turns out that we obtain a 'Black-like' option pricing formula:

$$c[r,t,T_1,T_2;K] = Z(r,t,T_2)N(d_+) - KZ(r,t,T_1)N(d_-) \tag{4.17}$$
$$p[r,t,T_1,T_2;K] = KZ(r,t,T_1)N(-d_-) - Z(r,t,T_2)N(-d_+) \tag{4.18}$$
$$\sigma_p = \sqrt{\beta}\sqrt{\frac{1-\exp(-2\gamma(T_1-t))}{2\gamma}}B(T_1,T_2) \tag{4.19}$$
$$d_\pm = \frac{\ln\frac{f(Z)}{K} \pm \frac{1}{2}\sigma_p^2}{\sigma_p} \tag{4.20}$$

Remarkably, the valuation of options on coupon-bearing bonds is also possible. Although the model was originally posited with the underlying stochastic process being Vasicek's model, it can be applied to any derivative where the value of zero-coupon bonds is dependent on the short rate only. The intuition is that a set of mini-options on each of the coupons and the bond can be constructed which will have the same value as the total option. Because the movements of the entire yield curve are perfectly correlated there is currently no additional value to having this decomposition. Thus, the value of the option on the coupon bearing bond can be decomposed into a portfolio of options on zero coupon bonds.

This procedure only works if the option is European.

Consider a European call with exercise price K and maturity T, on a coupon-bearing bond. Suppose that the bond provides cashflows $c_1,\ c_2,\ \dots,\ c_n$ at times $t_1,\ t_2,\ \dots,\ t_n$ after the maturity T of the option. The payoff of the option at time T is

$$\max\left[0, \sum_{i=1}^{n} c_i Z(r,T,t_i) - K\right] \tag{4.21}$$

where r is the short rate as observed at time T; of course unknown at time t. Let r^* be the value of the short rate at time T which causes the coupon-bearing bond price to equal the strike price. r^* is the solution of

$$K = \sum_{i=1}^{n} c_i Z(r^*,T,t_i) \tag{4.22}$$

We calculate r^* now using Newton's method. (Just using goalseek in excel will suffice for simple problems.) Since zero prices are decreasing functions of r, it follows that the option expires in the money for all $r < r^*$, and out of the money for all $r > r^*$. Then

$$\begin{aligned}
\text{payoff} &= \max\left[0, \sum_{i=1}^{n} c_i Z(r,T,t_i) - K\right] \\
&= \max\left[0, \sum_{i=1}^{n} c_i Z(r,T,t_i) - \sum_{i=1}^{n} c_i Z(r^*,T,t_i)\right] \\
&= \sum_{i=1}^{n} c_i \max[0, Z(r,T,t_i) - Z(r^*,T,t_i)]
\end{aligned}$$

Thus the call on the coupon-bearing bond is reduced to the sum of separate calls on the underlying zeros. This technique is called the 'Jamshidian trick'.

Example 4.4.1. Suppose that $\sigma = \sqrt{\beta} = 0.02$, $\gamma = 0.1$, $\theta = 0.01$ and the short rate be 10%. Consider a 3-year European put option on a 5-year bond with a coupon of 10% NACS, strike price $K = 98$, par 100.

At the end of 3 years there will be four cashflows remaining, namely

time	amount	discount factor
3.5	5	$\exp(A(3;3.50) - r(3)B(3;3.5))$
4	5	$\exp(A(3;4) - r(3)B(3;4))$
4.6	5	$\exp(A(3;4.5) - r(3)B(3;4.5))$
5	105	$\exp(A(3;5) - r(3)B(3;5))$

where the functional form of $A(t,T)$ and $B(t,T)$ are known - the only unknown here is $r(3)$, the value of the short rate that will actually be observed at time 3.

Solving (4.22) for $r^* = r(3)$, where $K = 98$ gives $r^* = 0.1095222$.

time	A	B	Z	Cash	Value
0	0	0	1.000000		
0.5	-0.001	0.4877	0.946830	5	4.734149
1	-0.005	0.9516	0.896731	5	4.483653
1.5	-0.011	1.3929	0.849538	5	4.247691
2	-0.018	1.8127	0.805091	105	84.534506
					98

The option is the sum of four options:

time	Cash	Strike
3.5	5	4.734149
4	5	4.483653
4.5	5	4.247691
5	105	84.53451

We value each as follows:

time	A	B	Z	Cash	Strike	σ_p	d_1	d_2	V
3.5	-0.052	2.9531	0.706252	5	4.734149	1.465%	0.376364	0.361713	0.012449
4	-0.067	3.2968	0.672465	5	4.483653	2.859%	0.3903	0.361713	0.022830
4.5	-0.083	3.6237	0.640436	5	4.247691	4.184%	0.403556	0.361713	0.031429
5	-0.101	3.9347	0.610074	105	84.53451	5.445%	0.416166	0.361713	0.808417
									0.875125

4.4.3 Cox-Ingersoll-Ross model

The Cox et al. [1985] model has all four parameters constant, but this time $\beta = 0$ and $\alpha > 0$. In this case (4.4) becomes:

$$dr = (\theta - \gamma r)dt + \sqrt{\alpha r}dz \tag{4.23}$$

The spot rate again has a mean reversion level of $\frac{\theta}{\gamma}$, with reversion occurring at a rate of γ but, unlike the Vasicek model, the steady-state distribution is not a special case of the normal distribution (since the volatility is now rate dependent). The CIR model does not permit negative interest rates, provided the technical condition $\theta > \frac{1}{2}\alpha$ holds. CIR give explicit solutions for some interest rate derivatives but the solutions are complicated and sometimes involve integrals that do not have exact solutions (and have to be estimated numerically).

For the pricing of discount bonds under the affine model (4.7) the bond pricing equations (4.13) and (4.12) reduce to:

$$\frac{dB}{dt} = \tfrac{1}{2}\alpha B^2 + \gamma B - 1$$
$$\frac{dA}{dt} = \theta B$$

which has the solution:

$$Z(t,T) = \frac{2(\exp(\psi_1(T-t)) - 1)}{(\gamma+\psi_1)(\exp(\psi_1(T-t)) - 1) + 2\psi_1}$$
$$\psi_1 = \sqrt{\gamma^2 + 2\alpha}$$
$$A(t,T) = \frac{2\theta}{\alpha} \ln \frac{2\psi_1 \exp((\gamma+\psi_1)(T-t)/2)}{(\psi_1+\gamma)(\exp(\psi_1(T-t)) - 1) + 2\psi_1}$$

For specific values of α, γ and θ we can generate yield curves of various shapes. Again, A and B are functions only of $T-t$. We can use a similar approach to Jamshidian's to value European style options on coupon-bearing bonds.

Consider a call option on an T_2-maturity discount bond with exercise price K and expiration $T_1 < T_2$. Using risk-neutral valuation, the vanilla option price is:

$$V_{\pm}[r,t,T_1,T_2;K] = Z(r,t;T_1)\mathbb{E}_t^{\mathbb{Q}}[\max(0, \pm(Z-K))]$$

where

$$f(Z) = \frac{Z(r,t,T_2)}{Z(r,t,T_1)}$$

Then,

$$V_c[r,t,T_1,T_2;K] = Z(r,t,T_1)\left[f(Z)\chi^2(h_1) - K\chi^2(h_2)\right]$$

where χ^2 is the non-central chi-squared distribution. The parameters h_1 and h_2 have a complicated dependence on α, γ and θ.

4.5 Two factor equilibrium models

In Brennan and Schwartz [1979], Brennan and Schwartz [1982] the process for the short rate reverts to a long rate, which is turn follows a stochastic process. The long rate is chosen as the yield on a perpetual bond.

Longstaff and Schwartz [1992a], Longstaff and Schwartz [1992b] starts with a general equilibrium model of the economy and derives a term structure model where is there is stochastic volatility. This model is analytically quite tractable.

4.6 Equilibrium models of the logarithm of the short rate

4.6.1 The Continuous time version of the Black-Derman-Toy model

Again, we can use Rebonato [1998] for the derivation of the continuous equivalent of the Black et al. [1990] model.

In the Black et al. [1990] model, a lognormal distribution of the short term interest rate is assumed i.e. $\ln r(t)$ is normally distributed. At each node $(i, j-1)$ we have 2 possible states of the world and interest rates denoted $r(i,j)$ and $r(i+1,j)$ that we can evolve to. The mean short term interest rate at this time may then be calculated as:

$$\ln r_m(i,j) = \tfrac{1}{2}\left[\ln r(i,j) + \ln r(i+1,j)\right]$$

Also

$$r(i+1,j) = r(i,j)\exp\left[2\sigma(j)\sqrt{\Delta t}\right]$$

and hence, we find each of the two possible rates as an offset from the median rate of interest, $r_m(i,j)$

$$\begin{aligned} r(i+1,j) &= r_m(i,j)\exp\left[\sigma(j)\sqrt{\Delta t}\right] \\ r(i,j) &= r_m(i,j)\exp\left[-\sigma(j)\sqrt{\Delta t}\right] \end{aligned}$$

So, the correct continuous analogue of this is as follows

$$r(t) = u(t)\exp\left(\sigma(t)z(t)\right) \tag{4.24}$$

where

$$\begin{aligned} u(t) &\sim \text{time } t \text{ median of the short term interest rate distribution,} \\ \sigma(t) &\sim \text{short term interest rate volatility at time } t, \\ z(t) &\sim \text{standard Brownian motion.} \end{aligned}$$

To examine the nature of the stochastic process driving the short term interest rate we must examine the evolution of $f(z,t) = \ln r(t) = \ln(u(t)) + \sigma(t)z(t)$. So

$$\begin{aligned} df &= d\,\ln u(t) + d(\sigma(t)z(t)) \\ &= \frac{\partial \ln u(t)}{\partial t}\,dt + \sigma'(t)z(t)\,dt + \sigma(t)\,dz(t) \\ &= \frac{u'(t)}{u(t)}\,dt + \sigma'(t)\frac{\ln r(t) - \ln u(t)}{\sigma(t)}dt + \sigma(t)\,dz(t) \\ &= \left[\frac{u'(t)}{u(t)} + \frac{\sigma'(t)}{\sigma(t)}(\ln r(t) - \ln u(t))\right]dt + \sigma(t)\,dz(t) \\ &= \frac{\sigma'(t)}{\sigma(t)}\left[\ln r(t) - \theta(t)\right]dt + \sigma(t)\,dz(t) \\ \theta(t) &:= -\frac{u'(t)}{u(t)}\frac{\sigma(t)}{\sigma'(t)} + \ln u(t) \end{aligned}$$

See [Cairns, 2004, §5.2.3].

It is tempting to work one-factor models in order to match not only the term structure but also to match the price of caps (liquid instruments) and swaptions. However, this amounts to over-parameterisation and leads to a non-stationary volatility structure. The corresponding volatility term structure implied by the model is unlikely to be anything like the existing volatility structure and can create derivative mispricing.

4.6.2 The Black-Karasinski Model

Black and Karasinski [1991] is the general version of the above:

$$d \ln r(t) = \alpha(t)(\ln \mu(t) - \ln r(t))dt + \sigma(t)dz(t) \tag{4.25}$$

See [Cairns, 2004, §5.2.3]. Now let $Y(t) = \ln r(t)$ in the equation above so that

$$dY(t) = \alpha(t)(\ln \mu(t) - Y(t))dt + \sigma(t)dz(t).$$

Then $r = e^Y$ so

$$\begin{aligned} dr &= e^Y\, dY + \tfrac{1}{2} e^Y \langle dY, dY \rangle \\ &= r(t)\left[\alpha(t)(\ln \mu(t) - Y(t))dt + \sigma(t)dz(t)\right] + \tfrac{1}{2} r(t)\sigma^2(t)\, dt \\ &= r(t)\left[\alpha(t)(\ln \mu(t) - \ln r(t)) + \tfrac{1}{2}\sigma^2(t)\right] dt + r(t)\sigma(t)dz(t) \end{aligned}$$

Alternatively, we can solve for the dynamics of Y. This time the integrating factor is $e^{A(t)}$ where $A(t) = \int_0^t \alpha(s)\, ds$. So let $f(y,t) = ye^{A(t)}$.

$$\begin{aligned} df(t) &= Y(t)e^{A(t)}\alpha(t)\, dt + e^{A(t)}\, dY \\ &= Y(t)e^{A(t)}\alpha(t)dt + e^{A(t)}\left[\alpha(t)(\ln \mu(t) - Y(t))dt + \sigma(t)dz(t)\right] \\ &= e^{A(t)}\left[\alpha(t) \ln \mu(t)dt + \sigma(t)dz(t)\right] \\ Y(t) &= e^{-A(t)}\left[Y(0) + \int_0^t e^{A(s)}\alpha(s) \ln \mu(s)ds + \int_0^t e^{A(s)}\sigma(s)dz(s)\right] \\ Y(t) &= e^{-A(t)}Y(0) + \int_0^t e^{A(s)-A(t)}\alpha(s) \ln \mu(s)ds + \int_0^t e^{A(s)-A(t)}\sigma(s)dz(s) \end{aligned}$$

which shows that

$$Y(t) \sim \phi\left(e^{-A(t)}Y(0) + \int_0^t e^{A(s)-A(t)}\alpha(s) \ln \mu(s)ds, \int_0^t e^{2(A(s)-A(t))}\sigma^2(s)\, ds\right)$$

This allows for a fairly rich calibration mechanism, using inter alia the caplet volatilities.

4.7 No-arbitrage models

We now consider no-arbitrage models. These will be generalisations of equilibrium models where the existing term structures are taken as input to the model and parameters are chosen so that those same term structures are outputs of the model.

4.7.1 The Continuous time version of the Ho-Lee model

Rebonato [1998] presents a simple analysis by which the continuous time equivalent of any discrete time model, modelled within a binomial lattice, may be found. See also [Svoboda, 2003, §10.4.1]. Given that we are in state $(i, j-1)$, we can only move in the next time step to state (i, j) or state $(i+1, j)$. Given the assumption that the short term interest rate follows a Gaussian process we have

$$r(i+1, j) = r(i, j) + 2\sigma(j)\sqrt{\Delta t} \tag{4.26}$$

where $\sigma(j)$ is the volatility of the one period rate that is earned in the period $[j\,\Delta t, (j+1)\,\Delta t]$. Let $r_m(i, j)$ be the expected interest rate at time j given where we are at time $j-1$, hence:

$$\begin{aligned} r_m(i,j) &= \tfrac{1}{2}\left[r(i+1,j) + r(i,j)\right] \\ \Rightarrow \quad r(i+1,j) &= r_m(i,j) + \sigma(j)\sqrt{\Delta t} \\ r(i,j) &= r_m(i,j) - \sigma(j)\sqrt{\Delta t} \end{aligned}$$

and so, in continuous time, we may write:

$$r(t) = \mu(t) + \sigma(t)z(t)$$

We may apply Itô's Lemma to determine the stochastic process for the short term interest rate, and if we assume that the volatility is also not time dependent then

$$\begin{aligned} dr &= \frac{\partial r}{\partial t}\,dt + \frac{\partial r}{\partial z}\,dz + \tfrac{1}{2}\frac{\partial^2 r}{\partial z^2}\,(dz)^2 \\ &= \frac{\partial r}{\partial t}\,dt + \sigma dz \end{aligned}$$

and so have our favoured affine form: we have $\beta > 0$ constant, $\alpha = 0 = \gamma$, but θ a function of time:

$$dr = \theta(t)dt + \sqrt{\beta}dz \tag{4.27}$$

There is no mean reversion in the model which means that the process for r has unbounded variance. The function $\theta(t)$ can be determined analytically and calibrated from the initial term structure of interest rates. (4.12) and (4.13) have become

$$\frac{\partial A}{\partial t} = \theta(t)B - \tfrac{1}{2}\beta B^2 \tag{4.28}$$

$$\frac{\partial B}{\partial t} = -1 \tag{4.29}$$

and so

$$B(t,T) = T - t \tag{4.30}$$

$$A(t,T) = -\int_t^T \theta(s)(T-s)\,ds + \tfrac{1}{6}\beta(T-t)^3 \tag{4.31}$$

We can calibrate $\theta(\cdot)$ so that the observed yield curve (found from our yield curve bootstrap) fits exactly. The best way to work with this is to think of t as today, so $t = 0$ and that the variable in play is T. Thus

$$-T\mathtt{r}(r, 0, T) = A(0,T) - rB(0,T)$$

We use (4.30) and (4.31) in this, and then differentiate twice w.r.t. T.[2] The result is

$$\theta(T) = f'(0,T) + \beta T \tag{4.32}$$

where $f(0,T)$ refers to the continuous forward rate for time T, as seen at time 0 (with the world in state $r = r(0)$). Of course, here the interpolated curve needs to be twice differentiable. Most cubic spline interpolation schemes satisfy this property, but of course there are plenty of problems with such interpolation schemes. The best approach might be to use the method of Svensson [1994].

Under the Ho-Lee model we can value European options on default-free bonds in exactly the same way as before. Consider a call option on an T_2-maturity discount bond with exercise price K and expiration $T_1 < T_2$. Using risk-neutral valuation, the vanilla option price is:

$$V_{\pm}[r,t,T_1,T_2;K] = Z(r,t,T_1)\mathbb{E}_t^{\mathbb{Q}}[\max(0,\pm(Z-K))]$$

where Z is the zero coupon price for expiry T_2 as observed at time T_1. It has current forward value

$$f(Z) = \frac{Z(r,t,T_2)}{Z(r,t,T_1)}.$$

It turns out that we obtain a 'Black-like' option pricing formula:

$$c[r,t,T_1,T_2;K] = Z(r,t,T_2)N(d_+) - KZ(r,t,T_1)N(d_-) \tag{4.33}$$

$$p[r,t,T_1,T_2;K] = KZ(r,t,T_1)N(-d_-) - Z(r,t,T_2)N(-d_+) \tag{4.34}$$

$$\sigma_p^2 = \beta(T_2-T_1)^2 T_1 \tag{4.35}$$

$$d_{\pm} = \frac{\ln\frac{f(Z)}{K} \pm \frac{1}{2}\sigma_p^2}{\sigma_p} \tag{4.36}$$

4.7.2 The Extensions of Hull & White

Hull and White [1990], Hull and White [1994] extended the Vasicek and CIR models to include time-dependency in the drift and volatility parameters. The extended Vasicek model is

$$dr = (\theta(t) - \gamma r(t))dt + \sqrt{\beta}dz \tag{4.37}$$

and the extended CIR model is

$$dr = (\theta(t) - \gamma r(t))dt + \sqrt{\alpha r(t)}dz \tag{4.38}$$

The extended Vasicek model with constant volatility is analytically tractable and results in a richer volatility structure for forward rates, spot rates and discount bonds.

(4.12) and (4.13) have become

$$\frac{\partial A}{\partial t} = \theta(t)B - \tfrac{1}{2}\beta B^2$$
$$\frac{\partial B}{\partial t} = -1 + \gamma B$$

[2] Here we are using the Leibnitz rule, for differentiation of a definite integral with respect to a parameter [National Institute of Standards and Technology, 2010, §1.5(iv)]:

$$\frac{d}{d\xi}\int_{\psi(\xi)}^{\phi(\xi)} f(\psi,\xi)\,d\psi = f(\phi(\xi),\xi)\frac{d\phi(\xi)}{d\xi} - f(\psi(\xi),\xi)\frac{d\psi(\xi)}{d\xi} + \int_{\psi(\xi)}^{\phi(\xi)} \frac{d}{d\xi}f(\psi,\xi)\,d\psi$$

and so

$$B(t,T) = \tfrac{1}{\gamma}(1 - e^{-\gamma(T-t)})\ ^{3}$$
$$A(t,T) = -\int_t^T \theta(s)B(s,T)\,ds + \tfrac{1}{2}\beta\int_t^T B(s,T)^2\,ds$$

In other words,

$$\int_0^T \theta(s)B(s,T)\,ds = \tfrac{1}{2}\beta\int_0^T B(s,T)^2\,ds - rB(0,T) + \mathbb{r}(r,0,T)T$$

We need to derive the function $\theta(T)$ at time $t = 0$, today.
We now differentiate twice w.r.t. T. Of course, we are able the plug in the known values of $B(s,T)$, $\frac{\partial}{\partial T}B(s,T)$ and $\frac{\partial^2}{\partial T^2}B(s,T)$. After some work, the result is

$$\theta(T) = f'(0,T) + \gamma f(0,T) + \frac{\beta}{2\gamma}(1 - e^{-2\gamma T}) \tag{4.39}$$

Option pricing formulae are the same as in §4.4.2. Note that θ was not a parameter in the option price there and $\theta(t)$ is not here - this information is reflected in the formulae through the zero coupon bond prices.

Calibrating the Hull-White (Vasicek) model

In order to calibrate this model, we still need to find the short rate volatility σ, and the mean reversion speed γ.

- Strip caps into caplets as in §3.4.
- As in §3.3.2, a caplet is actually a put on a bond.
- For any given value of σ and γ, the puts on bonds have prices as functions of β and γ and the current yield curve, which is given and fixed. These formulae appear earlier in this section, and refer back to the formulae of §4.4.2.
- We can then formulate a function an error function which measures the difference between model and market (as stripped) prices. This function will be something like

$$\mathrm{err}_{\sigma,\gamma} = \sum_i |V_i(\sigma,\gamma) - P_i| \tag{4.40}$$

- We minimise this function using the Nelder-Mead algorithm.

One will often find that a stable minimum is not easily found, and so one can fix one of the parameters (such as putting $\gamma = 0.03$; apparently the Bloomberg solution) and then solve for just the one free parameter σ.
These are also the parameters one will need to use in the market standard convexity adjustment that makes futures prices like forward prices.

[3] In showing this, one must be careful to distinguish the cases $\gamma B - 1 <> 0$.

4.7.3 The need for the convexity adjustment

The forward price of a contract is not the same as the futures price. As the underlying rate is positively correlated with spot interest rates, futures prices are higher than forward prices. To understand the nature of the bias between forward and futures contracts when the underlying is an interest rate, consider a money market futures contract and a forward rate agreement (FRA) on a certain forward rate.

An investor holding futures contracts realises the gains/losses resulting from a change in the position's value daily. On the other hand, an investor holding forward rate agreements realises the gains/losses only at the maturity of the underlying interest rate; the value of these gains/losses is the present value thereof. The gains realised on a long FRA position when the forward rate increases will be discounted at a higher interest rate. The losses realised on a long FRA position when the forward rate decreases will be discounted at a lower interest rate. This has the effect of decreasing the gains and increasing the losses for a long FRA position compared to the long futures position.

If the discount rate was unaffected by the change in the FRA rate, then a long position in a future could be perfectly hedged by a short position in $C(0, t)$ many positions in a FRA, where t is the commencement date of the FRA. However, as the spot rate (and hence the factor $C(0, t)$) is positively correlated with the FRA rate, the actual p&l of this position should be positive (in the simplest case of perfect correlation, the p&l is positive for all moves in the rate. See Figure 4.2.)

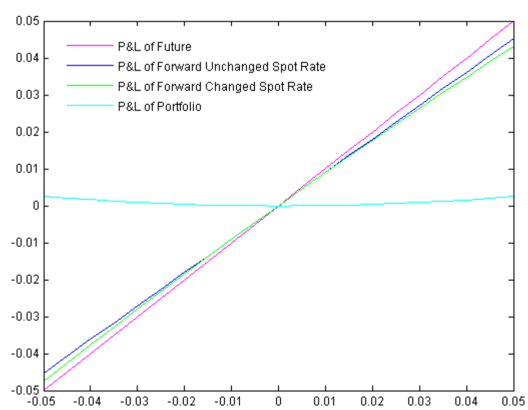


Figure 4.2: The p&l of a futures position hedged with short forwards, assuming the rates are equal and correlation between rates is perfect.

An alternative intuitive explanation for this phenomenon is as follows: the long party to the futures contract will tend to receive margin payments on days when interest rates rise, and make margin payments on days when interest rates decline. That investor will then invest profits they receive at higher interest rates, and fund losses they make at lower interest rates.

In either case we see that if futures and FRAs had the same rate, the future is far more attractive. Thus the fixed rate in the FRA should be lower in order to restore the attractiveness of the FRA.

However, recent research shows that this factor might not be taken into account in general i.e. that futures and forwards do not trade relative to each other in this way - see Poskitt [2008].

4.7.4 Derivation of the convexity adjustment

To bootstrap the current yield curve, forward rates are required. Typically, a futures rate will be input, an adjustment to make the rate 'forward like' will be made, and that rate will be used. This adjustment is model dependent. The models is use are the Ho-Lee model (perhaps the market standard) and the Hull-White model (the only one offered by Bloomberg, so will be the market standard soon enough). The difference between the futures rate and the forward rate is termed the *convexity adjustment*.

The theoretical futures rate $F(T_1, T_2)$, is calculated using the following equation

$$F(t; T_1, T_2) = \mathbb{E}_t^{\mathbb{Q}}[f(T_1; T_1, T_2)] \tag{4.41}$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the expectation under the risk-neutral measure and $f(t; T_1, T_2)$ is the forward rate at time t, for the time interval $[T_1, T_2]$. We can only solve this equation by introducing an interest rate model. These models will require some differentiability of the curve.

Ho-Lee Model

The derivation here is based on Hull.

The dynamics of the short rate under the Ho-Lee model are

$$dr(t) = \theta(t)dt + \sigma dW(t) \tag{4.42}$$

where σ is the volatility of the short rate, and $W(t)$ is standard Brownian motion under the risk-neutral measure. The $\theta(t)$ function is determined from the initial term structure as

$$\theta(t) = f'(0, t) + \sigma^2 t \tag{4.43}$$

where $f'(0, t)$ is the derivative with respect to t of the instantaneous forward rate for maturity t, as seen today. Under the Ho-Lee model, the price of a zero-coupon bond, $Z(t, T)$ is given by

$$Z(t, T) = e^{A(t,T) - r(t)(T-t)}$$

where

$$A(t, T) = \ln \frac{Z(0, T)}{Z(0, t)} - (T - t)\frac{\partial \ln Z(0, t)}{\partial t} - \frac{1}{2}\sigma^2 t(T - t)^2$$

Using Itô's lemma one shows that

$$\begin{aligned}
&dZ(t, T) \\
&= Z_t(t, T)dt + Z_r(t, T)dr(t) + \tfrac{1}{2}Z_{rr}(t, T)\left(dr(t)\right)^2 \\
&= \left[e^{A(t,T)}e^{-r(t)(T-t)}A_t(t, T) + e^{A(t,T)}e^{-r(t)(T-t)}r(t)\right]dt - e^{A(t,T)}e^{-r(t)(T-t)}(T - t)dr(t) \\
&\quad + \tfrac{1}{2}e^{A(t,T)}e^{-r(t)(T-t)}(T - t)^2\sigma^2 dt \\
&= Z(t, T)\left[A_t(t, T) + r(t)\right]dt - Z(t, T)(T - t)\left[\theta(t)dt + \sigma dW(t)\right] + \tfrac{1}{2}Z(t, T)(T - t)^2\sigma^2 dt \\
&= Z(t, T)\left[A_t(t, T) + r(t) + \tfrac{1}{2}(T - t)^2\sigma^2 - (T - t)\theta(t)\right]dt - Z(t, T)(T - t)\sigma dW(t)
\end{aligned}$$

and since

$$A_t(t,T) = (T-t)\theta(t) - \tfrac{1}{2}(T-t)^2\sigma^2 \tag{4.44}$$

it follows that

$$dZ(t,T) = r(t)Z(t,T)\,dt - \sigma Z(t,T)\,dW(t)$$

Now using (1.12) we can again use Itô's lemma to find

$$\begin{aligned}
&df(t;T_1,T_2)\\
&= f_t(t;T_1,T_2)dt + f_r(t;T_1,T_2)dr(t) + \tfrac{1}{2}f_{rr}(t;T_1,T_2)(dr(t))^2\\
&= \frac{1}{T_2-T_1}\left[\frac{A_t(t,T_1)Z(t,T_1)+r(t)Z(t,T_1)}{Z(t,T_1)} - \frac{A_t(t,T_2)Z(t,T_2)+r(t)Z(t,T_2)}{Z(t,T_2)}\right]dt\\
&+ \frac{1}{T_2-T_1}\left[-\frac{(T_1-t)Z(t,T_1)}{Z(t,T_1)} + \frac{(T_2-t)Z(t,T_2)}{Z(t,T_2)}\right](\theta(t)dt+\sigma dW(t)) + 0dt\\
&= \frac{1}{T_2-T_1}\left[A_t(t,T_1)-A_t(t,T_2)\right]dt + \theta(t)dt + \sigma dW(t)
\end{aligned}$$

and again using (4.44) we have that

$$\begin{aligned}
df(t;T_1,T_2) &= \left[\frac{A_t(t,T_1)-A_t(t,T_2)}{T_2-T_1} + \theta(t)\right]dt + \sigma\,dW(t)\\
&= \frac{\sigma^2}{2}(T_2+T_1-2t)\,dt + \sigma\,dW(t)
\end{aligned}$$

Thus

$$\begin{aligned}
F(0;T_1,T_2) &= \mathbb{E}_t^{\mathbb{Q}}[f(T_1;T_1,T_2)]\\
&= f(0;T_1,T_2) + \int_0^{T_1}\frac{\sigma^2}{2}(T_2+T_1-2s)\,ds\\
&= f(0;T_1,T_2) + \frac{1}{2}\sigma^2 T_1 T_2
\end{aligned}$$

Hull-White Model

In the Hull-White model

$$dZ(t,T) = r(t)Z(t,T)\,dt - B(t,T)\sigma Z(t,T)\,dW(t)$$

and after some work we get

$$F(0;T_1,T_2) = f(0;T_1,T_2) + \frac{B(T_2-T_1)}{T_2-T_1}\frac{\sigma^2}{4}\left[B(T_2-T_1)B(2T_1)+2B(T_1)^2\right]$$

4.8 Exercises

1. In the course of deriving the solution to the bond equation we arrive at two ordinary differential equations for the functions $A(t,T)$ and $B(t,T)$: equations (4.19) and (4,20). Assume $\alpha(t)$ and $\gamma(t)$ are constants. Integrate the equation (4.20) with boundary condition $B(T,T) = 0$ to obtain a closed form solution for $B(t,T)$.

2. Write down the form of the Bond-pricing equation when the short rate satisfies the following Vasicek model
$$dr = (\theta - \gamma r)\, dt + \sqrt{\beta} dz.$$
Solve the resulting ordinary differential equations for $A(t,T)$ and $B(t,T)$. Express the solution for $A(t,T)$ in terms of $B(t,T)$.

3. (a) What is the difference between an equilibrium short-term interest rate model and a no-arbitrage model?

   (b) If a stock price followed a mean-reverting or path-dependent process, there would be market inefficiency. Why is this **NOT** the case when short-term interest rate models are mean-reverting or path-dependent?

   (c) Suppose that the short rate is currently 3.5% and its volatility measure is 0.95% per annum. What happens to the volatility measure when the short rate increases to 10.5% in (i) Vasicek's model; (ii) Rendleman and Bartter's model; and (iii) the Cox, Ingersoll, Ross model?

4. (a) Given the following parametrisations of the Vasicek and CIR models:
$$\begin{aligned} dr &= (\theta - \gamma r)dt + \sqrt{\beta} dz \\ dr &= (\theta - \gamma r)dt + \sqrt{\alpha r}\, dz, \end{aligned}$$
respectively. Suppose that $\gamma = 0.15$ and $\theta = 0.015$, and that the initial short rate is $r = 10\%$. Let the initial volatility measure for the short rate be 2%. Compare the two different values given by the models for the 8-year discount factor. What is the associated 8-year, continuously-compounded interest rate in each instance?

   (b) What happens to the values above when $r = 9.99\%$? Calculate the present value of a one basis point shift ($PV01$) of a par 100 zero-coupon bond in each instance.

5. Perform the derivation of the $\theta(T)$ function in the Ho-Lee model.

6. (a) Suppose that $a = 0.25$, $b = 0.06$, and $\sigma = 0.022$ in Vasicek's model with the initial short-term interest rate being $r(0) = 5.8\%$. Calculate the price of a 2.20 year European call option on a coupon-bearing bond that will mature in three years' time. Suppose that the bond pays a coupon of 6% semi-annually. The par value of the bond is 100 and the strike price of the option is 99. The strike price is the cash price (not the quoted price) that will be paid for the bond.

   (b) Use the answer to (a) and put-call parity arguments to calculate the price of a put option that has the same terms as the call option.

(c) Plot the yield curve for these values of γ, b, σ and r out to 5 years.

7. Perform the derivation of the $\theta(T)$ function in the Hull-White-Vasicek model.

8. Suppose we are using Vasicek's model with a volatility of 4%, $\gamma = 15\%$, $\eta = 2\%$ and short rate $r = 10\%$.

   (a) Graph the yield curve.

   (b) Find the price of a one year call option on a three year zero coupon bond par 1, struck at 97c.

Chapter 5

The LIBOR market model

The input is a model for the LIBOR forward rates. These forward rates are of course extracted from the bootstrapped yield curve, while the volatilities are found from the cap/caplet Black model Black [1976] inputs.

In each Black model for maturity date T, the forward rates will be log-normally distributed in what is called the T-forward measure, where T is the pay date of the option. The fact that we can 'legally' discount the expected payoff under this measure using today's yield curve is a consequence of some profound academic work of Geman et al. [1995]. We can proceed to use Black's model without knowing any of the theory of the LMM (or of Geman et al. [1995]); however, the Black model cannot safely be used to value more complicated products where the payoff depends on observations at multiple dates.

The LIBOR market model is then a collection of Black models. Each of these models will have its own pricing measure. However, one can move from any one of these measures to any other by using a Radon-Nikodym change of measure. By changing all of the measures to a single measure (the one of furthest tenor) enables us to consider a single pricing measure.

In addition to the volatility inputs, the matrix of correlations between each of these forward rates is required - the correlations are the linkers between the forward rates. These correlations are not observable in the market. They either need to be determined from market data in the calibration step, or are given by exogenous considerations and then are inputs to further calibration routines. In the latter case, the fit to the market will be less satisfactory than in the former, but these models may subsequently perform in a more realistic fashion than methods where the fit is tighter.

The LIBOR market model is also commonly known as the BGM model after Brace et al. [1997]. The model was also initially developed by Miltersen et al. [1997] and Jamshidian [1997].

The LMM has become a standard in the banking industry, and is commonly used for pricing most exotic interest rate derivatives. Even in the case where other models would clearly be easier to use, the industry prefers to extend this model: possibly because it is a natural extension of the Black-Scholes world, and the amenability of Monte Carlo pricing for almost any product variation. Nevertheless, the ability of these models to capture rate curve dynamics is more than questionable Manuel Huyet [2007]: in reality there is clear evidence of jumps, regimes, and skew. The normality/lognormality assumption is of course quite questionable. The LMM is only now being extended to a regime which is skew aware e.g. Svoboda-Greenwood [2007], Rebonato [2007].

5.1 The model for a single forward rate

Let $L_i(t) := L(t; t_{i-1}, t_i)$ be the FRA-rate for the period from t_{i-1} to t_i. Clearly

$$C(t, t_i) = C(0, t_{i-1}) [1 + L_i(t)\alpha_i] \tag{5.1}$$

where t is today, $C(t, \cdot)$ are the capitalisation factors extracted from the yield curve, α_i is the time from t_{i-1} to t_i in years, observing the relevant day count convention. It follows that

$$L_i(t) = \frac{1}{\alpha_i}\left[\frac{Z(t, t_{i-1})}{Z(t, t_i)} - 1\right] \tag{5.2}$$

$$Z(t, t_i)L_i(t) = \frac{1}{\alpha_i}[Z(t, t_{i-1}) - Z(t, t_i)] \tag{5.3}$$

Of course this is just §1.7 again. In particular, $Z(t, t_i)L_i(t)$ is a tradeable asset. If we consider the measure for which $Z(t, t_i)$ is the numeraire - this is known as the t_i-forward measure - then $L_i(t)$ is a martingale under this measure. It now follows from the Martingale Representation Theorem[1]

$$dL_i(t) = L_i(t)v_i(t)' dW^i(t) \tag{5.4}$$

Here $W^i(t) = \begin{bmatrix} W_1^i(t) \\ W_2^i(t) \\ \vdots \\ W_n^i(t) \end{bmatrix}$ are correlated Brownian motions under the t_i-forward measure, and $v_i(t) = \begin{bmatrix} v_{1i}(t) \\ v_{2i}(t) \\ \vdots \\ v_{ni}(t) \end{bmatrix}$ are deterministic functions, and they are each 0 as soon as we reach t_{i-1}. In fact we can (and do) assume that all except the i^{th} entry of $v_i(t)$ are 0.

5.2 The pricing of caplets and caps

Since $v_i(t)$ is deterministic, we have

$$\Sigma^{t_i}[\ln L(t_{i-1})|\mathcal{F}_t] = \int_t^{t_{i-1}} \|v_i(s)\|^2\, ds$$

$$\mathbb{E}^{t_i}[\ln L(t_{i-1})|\mathcal{F}_t] = \ln L(t) - \tfrac{1}{2}\int_t^{t_{i-1}} \|v_i(s)\|^2\, ds$$

[1] This theorem tells you that that $dL_i(t) = v_i(t)' dW^i(t)$. Since $L_i(t)$ is never 0 this can be rewritten in the given form.

Thus the value of a caplet with strike K is given by

$$\begin{aligned}
V(t) &= Z(t,t_i)\alpha_i \mathbb{E}^{t_i}\left[(L(t_{i-1})-K)^+|\mathcal{F}_t\right] \\
&= Z(t,t_i)\alpha_i[L(t)N(d_1)-KN(d_2)] \\
d_\pm &= \frac{\ln\frac{L(t)}{K}\pm\frac{1}{2}\Sigma}{\sqrt{\Sigma}} \\
\Sigma &= \int_t^{t_{i-1}} \|v_i(s)\|^2\,ds
\end{aligned}$$

A cap is a collection of caplets. Thus the cap has value

$$\begin{aligned}
V(t) &= \sum_{i=1}^{M} Z(t,t_i)\alpha_i[L(t;t_{i-1},t_i)N(d_1^i)-KN(d_2^i)] \\
d_\pm^i &= \frac{\ln\frac{L(t;t_{i-1},t_i)}{K}\pm\frac{1}{2}\Sigma_i}{\sqrt{\Sigma_i}} \\
\Sigma_i &= \int_t^{t_{i-1}} \|v_i(s)\|^2\,ds
\end{aligned}$$

This shows that the LMM is in every way consistent with the Black model, and this model can recover all of the Black model inputs that are used for calibration.

5.3 A common measure

In §5.2 we could have used a single Brownian motion and a one dimensional $v(t)$. However, we want to price more complex derivatives based on many forward rates, possibly with many payoff times. So, we use a vector of Brownian motions.

We know numeraires for the measure t_{j-1} and for the 'next' measure t_j, so we can explicitly calculate the likelihood process:

$$\begin{aligned}
\eta(t) := \frac{d\mathbb{P}^{j-1}}{d\mathbb{P}^j}(t) &= \frac{Z(t,t_{j-1})}{Z(0,t_{j-1})}\Big/\frac{Z(t,t_j)}{Z(0,t_j)} \\
&= \frac{Z(0,t_j)}{Z(0,t_{j-1})}\frac{Z(t,t_{j-1})}{Z(t,t_j)} \\
&= \frac{Z(0,t_j)}{Z(0,t_{j-1})}(1+\alpha_j L_j(t))
\end{aligned}$$

Now

$$\begin{aligned}
d\eta(t) &= \frac{Z(0,t_j)}{Z(0,t_{j-1})}\alpha_j dL_j(t) \\
&= \frac{Z(0,t_j)}{Z(0,t_{j-1})}\alpha_j L_j(t) v_j(t)' dW^j(t) \\
&= \eta(t)\frac{\alpha_j L_j(t)}{1+\alpha_j L_j(t)} v_j(t)' dW^j(t)
\end{aligned}$$

and so by Girsanov's theorem

$$dW^{j-1}(t) = dW^j(t) - \frac{\alpha_j L_j(t)}{1+\alpha_j L_j(t)} v_j(t)\,dt$$

By induction,

$$dW^i(t) = dW^n(t) - \sum_{j=i+1}^{n} \frac{\alpha_j L_j(t)}{1+\alpha_j L_j(t)} v_j(t)\,dt$$

Note that the last equation is one of $n \times 1$ column vectors. It follows that

$$dL_i(t) = L_i(t) v_i(t)' \left[dW^n(t) - \sum_{j=i+1}^{n} \frac{\alpha_j L_j(t)}{1+\alpha_j L_j(t)} v_j(t)\,dt \right] \tag{5.5}$$

5.4 Pricing exotic instruments under LMM

In order to price derivatives, we now do the following

- Use Monte Carlo to evolve the forward rates to the first cash flow at t_i in the derivative.
- Calculate the cash flows in that experiment.
- Future value these cash flows forward to time t_n by using the capitalisation factors that have arisen from the experiment, NOT using today's forward capitalisation factor $C(t; t_i, t_n)$.
- Present value back to today by using today's discount factor $Z(t, t_n)$.
- Continue the Monte Carlo evolution to t_{i+1} and repeat.

5.4.1 A simple example

Suppose now is time t_0. In three months time (time t_1) we observe the LIBOR rate. If it is greater than 10.5%, a caplet on the then three month rate (to be observed at time t_2) is knocked in with a strike of 10%. The payoff occurs in advance, 6 months from now. 9 months from now is time t_3. Let L_0 be the current LIBOR rate, L_1 be the rate for the 3x6 period, and L_2 the rate for the 6x9 period. Let v_1 and v_2 be the volatilities and ρ the correlation.

$$\begin{aligned} dL_1(t) &= L_1(t) v_1(t)' dW^1(t) \\ dL_2(t) &= L_2(t) v_2(t)' dW^2(t) \end{aligned}$$

Also

$$\begin{aligned} dW^1(t) &= dW^2(t) - \frac{\alpha_2 L_2(t)}{1+\alpha_2 L_2(t)} v_2(t)\,dt \\ \Rightarrow dL_1(t) &= L_1(t) v_1(t)' \left[dW^2(t) - \frac{\alpha_2 L_2(t)}{1+\alpha_2 L_2(t)} v_2(t)\,dt \right] \end{aligned}$$

Thus

$$dL_1(t) = L_1(t)v_{11}(t)dW_1^2(t) - L_1(t)\frac{\alpha_2 L_2(t)}{1+\alpha_2 L_2(t)}\rho v_{11}(t)v_{22}(t)\,dt$$
$$dL_2(t) = L_2(t)v_{22}(t)dW_2^2(t)$$
$$\rho\,dt = dW_1^2(t)dW_2^2(t)$$

To find a solution of this scheme, we can proceed as follows: we find dW_1^2 and dW_2^2 using the Cholesky decomposition.[2] We then calculate dL_1 and dL_2 and the new values of L_1 and L_2 (we are now at time t_1). If $L_1 < 10.5\%$ we can exit - the option expires worthless. If not, we generate another dW_2^2 and another dL_2 and the new value of L_2 (we are now at time t_2). We then determine the payoff of the caplet with the strike of 10%. The payoff is in advance, so we capitalise the payoff with the observed value of L_2 to time t_3.

We take averages in our Monte Carlo experiment, and discount to today using the current 9m discount factor.

We have used the most naïve discretisation of the scheme. In reality, smaller time steps need to be taken. For details see [Brigo and Mercurio, 2006, §6.10].

5.4.2 Calibration of the parameters

- A decent yield curve bootstrap algorithm needs to be used. Such issues have already been discussed.
- Volatilities need to be determined from information in the market. This is the same cap to caplet problem seen in §3.4.
- Correlations need to be modelled.
- For the Monte Carlo we need low discrepancy sequences in high dimensions. Sobol' sequences are most suitable here, see [Jäckel, 2002, Chapter 8].

[2]Suppose there are two underlyings. Using excel/vba, we first extract pairs of uniformly distributed random numbers U_1, U_2, then transform them into pairs of independent normally distributed random numbers Z_1, Z_2 by using the inverse of the cumulative normal distribution. We then apply the Cholesky decomposition:

$$W_1 = Z_1, W_2 = \rho Z_1 + \sqrt{1-\rho^2}Z_2 \tag{5.6}$$

Now W_1 and W_2 are normally distributed random numbers with correlation ρ.

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